OPTIMAL CONTROL PROBLEM FOR A NONLINEAR DRIFTLESS 5-DIMENSIONAL SYSTEM WITH 2 INPUTS

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Abstract: We study the quasihomogeneous nilpotent approximation of a nonlinear 5-dimensional system linear in 2 control parameters and the maximal growth vector (2,3,5). For such a system we study optimal control problem with a quadratic cost. In geometry, it corresponds to the flat (2,3,5) sub-Riemannian structure.
In robotics, it appears as nilpotent approximation to the following systems: (1) a pair of bodies rolling one on another without slipping and twisting, (2) a car with 2 off-hooked trailers.
A characteristic feature of the problem is the presence of abnormal minimizers.
We study geodesics and the exponential mapping for the optimal control problem.
We analyze the conjugate locus in the neighborhood of abnormal minimizers.

Keywords: optimal control, nonlinear control systems

1. PROBLEM STATEMENT

We study the optimal control problem determined by the flat (2,3,5) sub-Riemannian structure.

Let L be the 5-dimensional nilpotent Lie algebra with the following commutation rules in some basis $X_1, \dots, X_5$:

$$[X_1, X_2] = X_3, \; [X_1, X_3] = X_4, \; [X_2, X_3] = X_5,$$
$$\operatorname{ad} X_4 = \operatorname{ad} X_5 = 0,$$

and let M be the connected, simply connected Lie group with the Lie algebra L. We consider $X_1, \dots, X_5$ as left-invariant vector fields on M.

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The pair of vector fields X_1, X_2 determines a left-invariant sub-Riemannian structure with the growth vector (2,3,5) on M:

$$\Delta = \operatorname{span}(X_1, X_2), \quad \langle X_i, X_j \rangle = \delta_{ij}, \; i, j = 1, 2,$$

called the flat (2,3,5) sub-Riemannian structure. Such a structure is unique, up to isomorphism of Lie groups. We are interested in sub-Riemannian length minimizers for this sub-Riemannian structure, i.e., in solutions to the following optimal control problem:

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \quad q \in M, \quad u_1, \; u_2 \in \mathbf{R}, \tag{1}$$

$$q(0) = q_0, \quad q(T) = q_1 \text{ fixed}, \tag{2}$$

$$l = \int_0^T \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{3}$$

Without loss of generality, we take the identity element of the Lie group M as the initial point q_0.

Problem (1)–(3) was considered by Brockett and Dai (1993).

2. MOTIVATION

The flat (2,3,5) sub-Riemannian structure gives a local nilpotent approximation for an arbitrary sub-Riemannian structure with the growth vector (2, 3, 5), see Agrachev and Sarychev (1988), Agrachev et al. (1989), Bellaiche (1996), and it is important both from theoretical and applied points of view.

The growth vector (2,3,5) is maximal, thus a generic rank 2 sub-Riemannian structure on a 5-dimensional manifold has the growth vector (2,3,5) at a generic point.

The preceding maximum growth case (2,3), i.e., the contact case, was already studied in detail:

- the flat (2,3) case evolving on the Heisenberg group was analyzed by Brockett (1981), and Vershik and Gershkovich (1987),
- the general (2,3) case was studied as a perturbation of the flat case by Agrachev (1996) and El-Alaoui et al. (1996).

The next maximum growth (thus generic) case is the growth vector (2,3,5).

In robotics, (2,3,5) sub-Riemannian structures appear as models of the following systems:

- a pair of bodies rolling one on another without slipping and twisting Li and Canny (1990); Bicchi et al. (1995); Agrachev and Sachkov (1999); Marigo and Bicchi (1999),
- a car with 2 off-hooked trailers Laumond (1998); Vendittelli et al. (1999).

3. MODEL

We choose the following model for the flat (2,3,5) sub-Riemannian structure:

$$M = \mathbf{R}^5_{x,y,z,v,w},$$
$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2+y^2}{2}\frac{\partial}{\partial w},$$
$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v}.$$

4. EXISTENCE OF SOLUTIONS

Control system (1) has the full rank and the state space M is connected, thus the system is globally controllable on M.

Existence of optimal controls in the optimal control problem (1)–(3) follows from the classical Filippov's theorem.

5. EXTREMALS

We replace the length functional (3) by action:

$$J = \frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt \to \min \tag{4}$$

and seek for extremals of the problem (1), (4) via Pontryagin Maximum Principle.

5.1 *Normal extremals*

Introduce the linear Hamiltonians corresponding to the basis fields:

$$h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \lambda \in T^*M, \quad i = 1, \dots, 5.$$

Normal extremals are trajectories of the Hamiltonian system

$$\dot{\lambda} = \vec{H}(\lambda), \quad \lambda \in T^*M, \tag{5}$$

with the normal Hamiltonian $H = (h_1^2 + h_2^2)/2$. In the coordinates h_i on vertical fibers in T^*M, system (5) reads

$$\dot{h}_1 = -h_2 h_3, \tag{6}$$
$$\dot{h}_2 = h_1 h_3, \tag{7}$$
$$\dot{h}_3 = h_1 h_4 + h_2 h_5, \tag{8}$$
$$\dot{h}_4 = 0, \tag{9}$$
$$\dot{h}_5 = 0, \tag{10}$$
$$\dot{q} = h_1 X_1 + h_2 X_2.$$

We consider geodesics parametrized by arc-length, i.e., restrict to the level surface $\{H = 1/2\}$. Using the polar coordinates $h_1 = \cos\theta$, $h_2 = \sin\theta$, $h_4 = \alpha \sin\beta$, $h_5 = -\alpha\cos\beta$, we reduce equations (6)–(10) to the mathematical pendulum equation:

$$\ddot{\theta} = -\alpha \sin(\theta - \beta), \quad \alpha,\ \beta = \text{const}, \tag{11}$$

which is known to be integrable in Jacobi elliptic functions. We find explicitly $\theta(t)$, $h_i(t)$, and finally $q(t)$ in the model in $\mathbf{R}^5$ in terms of elliptic functions cn, sn, cn, E.

5.2 *Abnormal extremals*

Abnormal geodesics are trajectories of the ODEs

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad u_1,\ u_2 = \text{const},$$

i.e., one-parameter subgroups tangent to the distribution Δ. The corresponding controls $u_1 = \cos\theta$, $u_2 = \sin\theta$, $\theta = \text{const}$, satisfy the pendulum equation (11), thus abnormal geodesics are not strictly abnormal.

In the model in $\mathbf{R}^5$, projections of the abnormal geodesics to the plane (x,y) are straight lines, thus abnormal geodesics $q(t)$ are optimal for $t \in [0,+\infty)$.

6. OPTIMALITY OF NORMAL GEODESICS

Small pieces of geodesics are optimal, so any geodesic $q(t)$ has a cut point, i.e., the first point where the geodesic fails to be optimal. At cut points, the global optimality of geodesics is lost. The local counterpart of cut points are conjugate points, where geodesics lose the local optimality.

Conjugate points are critical values of the exponential mapping:

$\mathrm{Exp} \,:\, (\lambda_0,t) \in C_0 \times \mathbf{R}_+ \mapsto q(t) = \pi \circ e^{t\vec{H}}\lambda_0 \in M,$

where $C_0 = \{H = 1/2\} \cap T^*_{q_0}M$ and $\pi : T^*M \to M$ is the projection.

The set of all first conjugate points to the point q_0 along all geodesics is denoted by Con_{q_0} and is called the conjugate locus.

In order to reduce dimensions, we use symmetries of the problem.

7. SYMMETRIES

It was known since the work of Cartan (1910) that infinitesimal symmetries of the flat (2,3,5) distribution Δ is the 14-dimensional exceptional simple Lie algebra $\mathfrak{g}_2$, see also Sachkov (1998). The Lie algebra of symmetries of the flat sub-Riemannian structure is 6-dimensional: it contains 5 "trivial" symmetries given by left translations on the Lie group M, and one rotation X_0 on M, $X_0(q_0) = 0$. We have

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1,$$

thus the field X_0 preserves parametrized geodesics starting from q_0:

$$e^{sX_0}q(t) = q'(t).$$

In the model, we have

$$X_0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w}.$$

There is also a symmetry Y of the distribution Δ which acts as homothety on the sub-Riemannian structure:

$$[Y, X_1] = -X_1, \quad [Y, X_2] = -X_2,$$

thus preserves nonparametrized geodesics but not time along them:

$$e^{rY}q(t) = q'(t'), \quad t' = e^{2r}t.$$

In the model,

$$Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 3v\frac{\partial}{\partial v} + 3w\frac{\partial}{\partial w}.$$

The symmetries X_0 and Y commute:

$$[X_0, Y] = 0.$$

8. FACTORIZATION

We use the symmetries X_0 and Y to factorize the exponential mapping and conjugate locus.

For the Hamiltonian lifts $\vec{h}_0$ and $\vec{h}_Y$ of the fields X_0 and Y respectively, we have:

$$[\vec{h}_0, \vec{H}] = 0, \qquad \vec{h}_0 H = 0,$$
$$[\vec{h}_Y, \vec{H}] = -2\vec{H}, \qquad \vec{h}_Y H = -2H,$$
$$[\vec{h}_0, \vec{h}_Y] = 0.$$

In order to preserve level sets of H, we replace the field $\vec{h}_Y$ by $Z = \vec{H}_Y + e$, where $e = \sum_{i=1}^5 h_i \frac{\partial}{\partial h_i}$ is the Euler vertical field on T^*M, so that

$$[Z, \vec{H}] = -\vec{H}, \quad ZH = 0, \quad [Z, \vec{h}_0] = 0.$$

We have

$$\mathrm{Exp}(e^{rZ}e^{s\vec{h}_0}\lambda_0, t) = e^{rY}e^{sX_0}\,\mathrm{Exp}(\lambda_0, te^r),$$

thus we can factorize the domain $N = C_0 \times \mathbf{R}_+$ of the exponential mapping and the manifold M and obtain a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\mathrm{Exp}} & M & \xleftarrow{\mathrm{Id}} & \mathrm{Con}_{q_0} \\ \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\pi_2} \\ N' & \xrightarrow{\mathrm{Exp}'} & M' & \xleftarrow{\mathrm{Id}} & \mathrm{Con}'_{q_0} \end{array}$$

Here $\pi_1 : N \to N' = N/e^{\mathbf{R}\vec{h}_0}e^{\mathbf{R}Z}$ and $\pi_2 : M \to M' = M/e^{\mathbf{R}X_0}e^{\mathbf{R}Y}$ are projections and Con'_{q_0} is the set of critical values of the factorized exponential mapping Exp'.

The action of the 2-parameter group of symmetries $e^{\mathbf{R}\vec{h}_0}e^{\mathbf{R}Z}$ reduces values of the parameters: $\alpha = 1$, $\beta = 0$, and equation (11) reduces to the standard pendulum equation:

$$\ddot{\theta} = -\sin\theta. \qquad (12)$$

After factorization, the set of geodesics $q'(\cdot)$ is parametrized by the phase portrait of the standard pendulum (12).

Abnormal geodesics correspond to equilibria of the pendulum: $(\theta, \dot{\theta}) = (0,0)$ and $(\theta, \dot{\theta}) = (\pi, 0)$.

9. FACTORIZED CONJUGATE LOCUS NEAR THE STABLE EQUILIBRIUM

We study the local structure of the factorized conjugate locus Con'_{q_0} in the neighborhood of the stable equilibrium of the pendulum: $(\theta, \dot{\theta}) = (0,0)$. The conjugate time is bounded. Asymptotically, the conjugate locus is a quasihomogeneous surface of orders (1, 1, 2), its intersection with a plane transversal to the axis is an astroid.

We present below the graph of the conjugate time (in the domain of Exp') and the conjugate locus Con'_{q_0} (in the range of Exp').

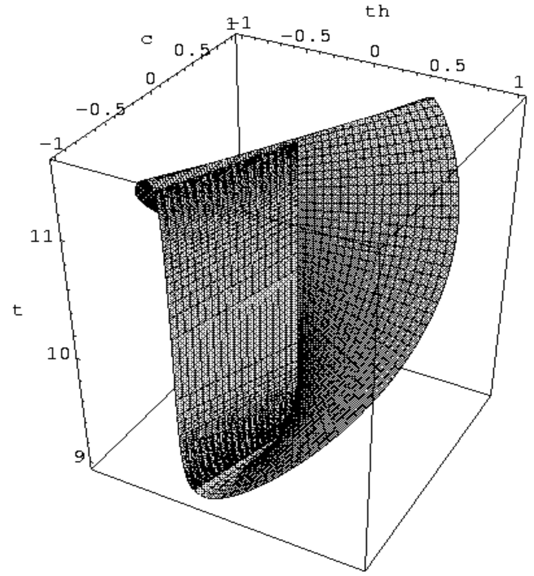

Figure 1: The conjugate time.

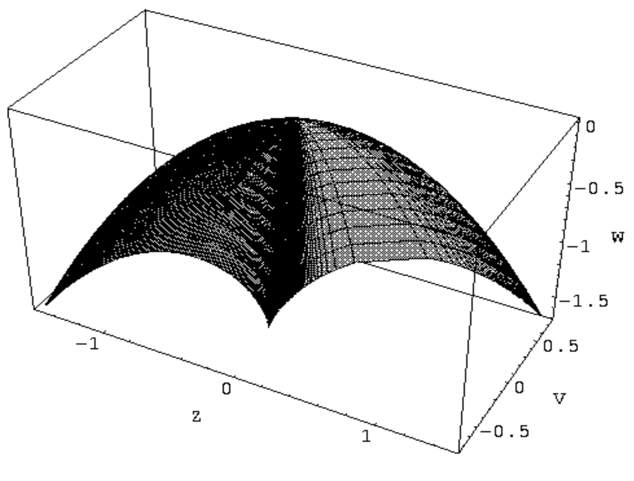

Figure 2: Con'_{q_0}.

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