Maxwell time and conjugate time in sub-Riemannian problem on the Engel group

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Problem Statement

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix},$$

$$q = (x, y, z, v) \in \mathbb{R}^4, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$q(0) = q_0 = (0,0,0,0)^T, \quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1)^T,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min \iff \int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min\,.$$

Overview

- Parameterization of extremal curves.
- Symmetries of exponential mapping and construction of the Maxwell sets.
- Global bound of the cut time.
- Local optimality and estimation of the conjugate time.
- Algorithm and software for numerical solution of the problem.

Known results for invariant sub-Riemannian problems on Lie groups

1. Three-dimensional Lie groups:
   - Heisenberg group (A. M. Vershik, V. Ya. Gershkovich 1986),
   - $SL(2), SO(3), S^3$ (U. Boscain, F. Rossi 2008),
   - $SE(2)$ (Yu. L. Sachkov 2010)
2. 5-dimensional nilpotent Lie group with growth vector $(2, 3, 5)$ (Yu.L.Sachkov 2006).
3. 6-dimensional nilpotent Lie group with growth vector $(3, 6)$ (O.M. Myasnichenko 2002).

Nilpotent sub-Riemannian problem on the Engel group

$$X_1 = (1, 0, -\frac{y}{2}, 0)^T, \qquad X_2 = (0, 1, \frac{x}{2}, \frac{x^2+y^2}{2})^T.$$

$$\begin{aligned}
&\mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, X_2, X_3, X_4),\\
&\dim \mathrm{Lie}(X_1, X_2)(q) = 4,\\
&[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,\\
&[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.
\end{aligned}$$

Growth vector (2, 3, 4).

Nilpotent approximation of nonholonomic control systems in four-dimensional space with two-dimensional control (e. g. car with trailer).

Controllability and existence of optimal curves

1. $X_1(q), \dots, X_4(q)$ are linearly independent $\forall q \in \mathbb{R}^4 \xrightarrow{\text{Rashevskii–Chow theorem}}$ complete controllability.

2. Existence of optimal solutions is implied by Filippov theorem.

Normal Hamiltonian system

$$\begin{aligned}
\dot\theta &= c, & \theta &\in S^1,\\
\dot c &= -\alpha \sin\theta, & c &\in \mathbb{R},\\
\dot\alpha &= 0, & \alpha &\in \mathbb{R},\\
\dot q &= \cos\theta X_1(q) + \sin\theta X_2(q).
\end{aligned}$$

$$E = \frac{c^2}{2} - \alpha\cos\theta \in [-|\alpha|, +\infty).$$

Equation of pendulum and physical meaning of α

$$\ddot{\theta} = -\alpha \sin\theta, \qquad \alpha = \frac{g}{L} = \text{const} \in \mathbb{R}$$

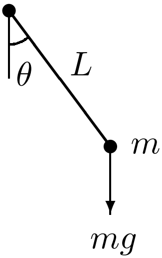


Figure: Mathematical pendulum with $\alpha > 0$

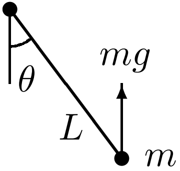


Figure: Mathematical pendulum with $\alpha < 0$

Stratification of phase cylinder of pendulum

$$C = T^*_{q_0} M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\}.$$

$$C = \cup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset,\ i \neq j.$$

$$C_i^+ = C_i \cap \{\alpha > 0\}, \qquad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\},$$

$$C_{i+}^{\pm} = C_i^{\pm} \cap \{c > 0\}, \qquad C_{i-}^{\pm} = C_i^{\pm} \cap \{c < 0\}, \qquad i \in \{2, 3\}.$$

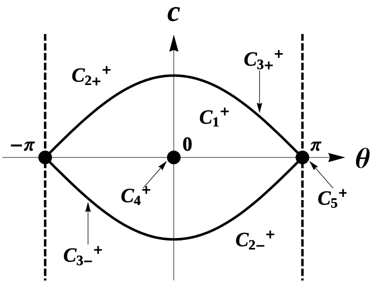


Figure: Stratification for $\alpha > 0$

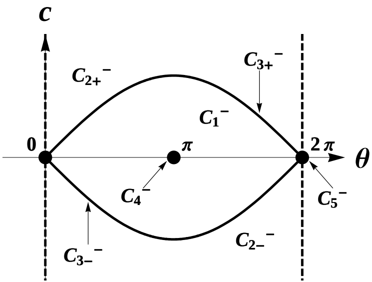
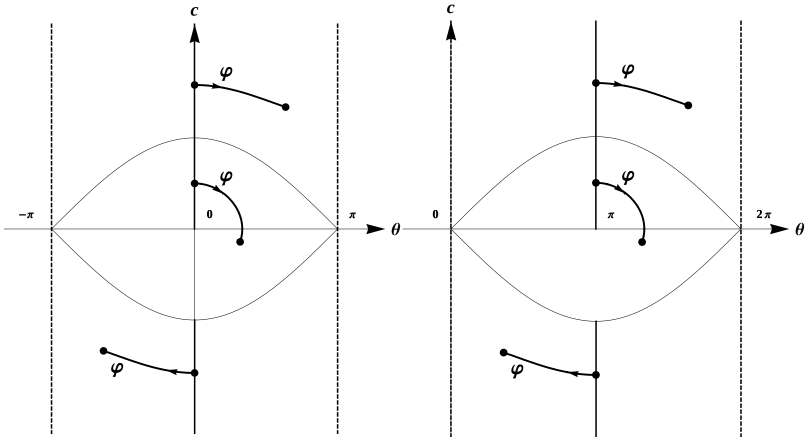


Figure: Stratification for $\alpha < 0$

Elliptic coordinates (φ, k) in the phase cylinder of pendulum



$$\dot{\varphi} = 1, \quad \dot{k} = 0, \quad \dot{\alpha} = 0.$$

Parametrization of extremal curves in the case $\alpha = 1$

$\lambda \in C_1^+$ (oscillations of pendulum) $\Rightarrow$

$$
\begin{aligned}
x_t &= 2k(\operatorname{cn}\varphi_t - \operatorname{cn}\varphi), \\
y_t &= 2(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) - t, \\
z_t &= 2k\left(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi - \frac{y_t}{2}(\operatorname{cn}\varphi_t + \operatorname{cn}\varphi)\right), \\
v_t &= \frac{y_t^3}{6} + 2k^2 \operatorname{cn}^2\varphi y_t - 4k^2 \operatorname{cn}\varphi(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi) + \\
&\quad + 2k^2\left(\frac{2}{3}\operatorname{cn}\varphi_t \operatorname{dn}\varphi_t \operatorname{sn}\varphi_t - \frac{2}{3}\operatorname{cn}\varphi \operatorname{dn}\varphi \operatorname{sn}\varphi + \frac{1-k^2}{3k^2} t + \right. \\
&\quad \left. \frac{2k^2-1}{3k^2}(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi))\right).
\end{aligned}
$$

Symmetries of Hamiltonian system

Dilation of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta, \frac{c}{\sqrt{\alpha}}, 1, \sqrt{\alpha}x, \sqrt{\alpha}y, \alpha z, \alpha^{\frac{3}{2}}v, \sqrt{\alpha}t),$$

$$(\varphi, k, \alpha) \mapsto (\sqrt{\alpha}\varphi, k, 1).$$

Inversion of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t),$$

$$(\varphi, k, \alpha) \mapsto (\varphi, k, -\alpha).$$

Parametrization of extremal trajectories in general case with $\lambda \in \cup_{i=1}^{3} C_i$

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = (\frac{s_1}{\sigma} x_{\sigma t}, \frac{s_1}{\sigma} y_{\sigma t}, \frac{1}{\sigma^2} z_{\sigma t}, \frac{s_1}{\sigma^3} v_{\sigma t})(\sigma\varphi, k, 1),$$

where $\sigma = \sqrt{|\alpha|}, s_1 = \operatorname{sgn} \alpha$.

General case with $\alpha \neq 0$

$$\lambda \in C_1 \Rightarrow$$

$$x_t = \frac{2k\sigma}{\alpha}(\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)),$$

$$y_t = \frac{2\sigma}{\alpha}(\operatorname{E}(\sigma\varphi_t) - \operatorname{E}(\sigma\varphi)) - \operatorname{sgn}\alpha t,$$

$$z_t = \frac{2k}{|\alpha|}(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi) -$$

$$- \frac{\sigma k y_t}{2\alpha}(\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi))),$$

$$v_t = \ldots$$

Parametrization of extremal curves for degenerate cases

$$\lambda \in C_4 \ \Rightarrow \ x_t = 0, \quad y_t = t \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = \frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_5 \ \Rightarrow \ x_t = 0, \quad y_t = -t \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = -\frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_6 \ \Rightarrow$$

$$x_t = \frac{\cos(ct+\theta) - \cos\theta}{c}, \quad y_t = \frac{\sin(ct+\theta) - \sin\theta}{c},$$

$$z_t = \frac{ct - \sin(ct)}{2c^2}, \quad v_t = -\frac{2c\cos\theta\, t - 4\sin(ct+\theta) + \sin(2ct+\theta)}{4c^3}.$$

$$\lambda \in C_7 \ \Rightarrow \ x_t = -t \sin\theta, \quad y_t = t \cos\theta, \quad z_t = 0, \quad v_t = \frac{\cos\theta}{6} t^3.$$

Exponential mapping, Maxwell points and cut time

$$\text{Exp} : C \times \mathbb{R}_+ \to M = \mathbb{R}^4,$$
$$\text{Exp}(\lambda, t) = q_t = (x_t, y_t, z_t, v_t),$$
$$\lambda = (\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+, \quad q_t \in M.$$

$$\text{MAX} = \{(\lambda, t) \mid \exists \tilde{\lambda} \neq \lambda, \text{Exp}(\lambda, t) = \text{Exp}(\tilde{\lambda}, t)\},$$

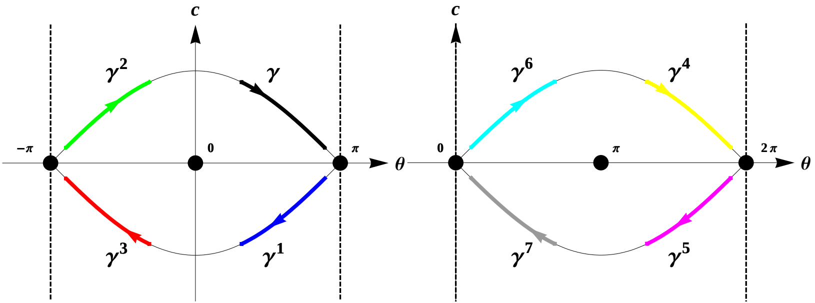
$$t_{cut}(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is optimal for } s \in [0, t]\},$$
$$t_{cut}(\lambda) \leq t \text{ for any } (\lambda, t) \in \text{MAX}.$$

Group of symmetries of exponential mapping

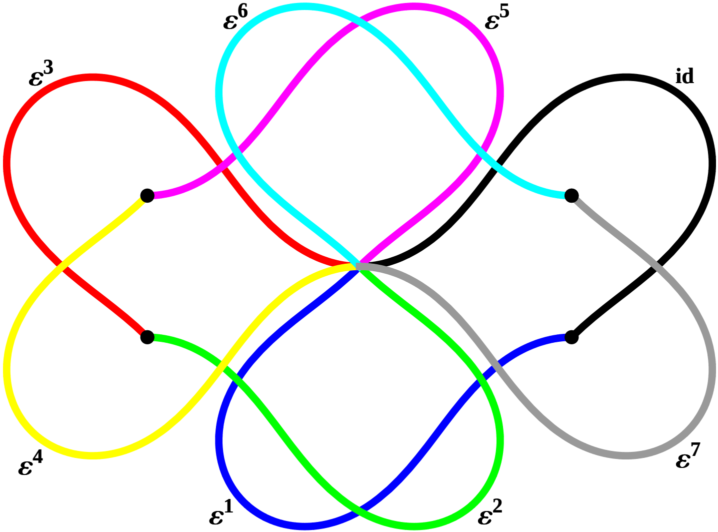
G	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
ε^1	Id	ε^3	ε^2	ε^5	ε^4	ε^7	ε^6
ε^2		Id	ε^1	ε^6	ε^7	ε^4	ε^5
ε^3			Id	ε^7	ε^6	ε^5	ε^4
ε^4				Id	ε^1	ε^2	ε^3
ε^5					Id	ε^3	ε^2
ε^6						Id	ε^1
ε^7							Id

Table: Multiplication in $G = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6, \varepsilon^7\}$

Reflections of trajectories of pendulum



Reflections of Euler elasticae



Reflections as symmetries of Exp

Proposition

Reflection ε^i is a symmetry of exponential mapping for any $i = 1, \dots, 7$, i. e.,

$$\varepsilon^i \circ \mathrm{Exp}(\theta, c, \alpha, t) = \mathrm{Exp} \circ \varepsilon^i(\theta, c, \alpha, t),$$
$$(\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+.$$

$$\mathrm{MAX}^i = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t)\},$$
$$\lambda = (\theta, c, \alpha), \quad \lambda^i = (\theta^i, c^i, \alpha^i) = \varepsilon^i(\lambda).$$

Fixed points of ε^i in the image of exponential mapping

$$\mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t) \iff \varepsilon^i(q_t) = q_t.$$

Lemma

1. $\varepsilon^1(q) = q \iff z = 0$,
2. $\varepsilon^2(q) = q \iff x = 0$,
3. $\varepsilon^3(q) = q \iff x^2 + z^2 = 0$,
4. $\varepsilon^4(q) = q \iff x^2 + y^2 + v^2 = 0$,
5. $\varepsilon^5(q) = q \iff x^2 + y^2 + z^2 + v^2 = 0$,
6. $\varepsilon^6(q) = q \iff y^2 + (2v - xz)^2 = 0$,
7. $\varepsilon^7(q) = q \iff y^2 + z^2 + v^2 = 0$.

Fixed points of ε^i in the preimage of exponential mapping

Proposition

If $(\lambda, t) \in C \times \mathbb{R}_+$, $\varepsilon^i(\lambda, t) = (\lambda^i, t)$ *then:*

1. $\lambda^1 = \lambda \iff \begin{cases} \operatorname{cn} \tau = 0 \text{ if } \lambda \in C_1 \\ \text{is impossible if } \lambda \in C_2 \cup C_3 \cup C_6 \end{cases}$
2. $\lambda^2 = \lambda \iff \begin{cases} \operatorname{sn} \tau = 0 \text{ if } \lambda \in C_1 \\ \operatorname{sn} \tau \operatorname{cn} \tau = 0 \text{ if } \lambda \in C_2 \\ \tau = 0 \text{ if } \lambda \in C_3 \\ 2\theta + ct = 2\pi n \text{ if } \lambda \in C_6 \end{cases}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2k}.$$

Complete description of the Maxwell sets for $\varepsilon^1, \varepsilon^2$

Theorem

1. $\mathrm{MAX}^1 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = p_z^n(k), n \in \mathbb{N}, \mathrm{cn}(\tau) \neq 0\}$,
2. $\mathrm{MAX}^1 \cap N_2 = \mathrm{MAX}^1 \cap N_3 = \mathrm{MAX}^1 \cap N_6 = \emptyset$,
3. $\mathrm{MAX}^2 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = 2Kn, n \in \mathbb{N}, \mathrm{sn}(\tau) \neq 0\}$,
4. $\mathrm{MAX}^2 \cap N_2 = \{(\lambda, t) \in N_2 \mid p = Kn, n \in \mathbb{N}, \mathrm{sn}(\tau)\,\mathrm{cn}(\tau) \neq 0\}$,
5. $\mathrm{MAX}^2 \cap N_3 = \emptyset$,
6. $\mathrm{MAX}^2 \cap N_6 = \{(\lambda, t) \in N_6 \mid tc = 2\pi n, \theta \neq \pi k, n, k \in \mathbb{Z}\}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2k}.$$

$p_z^n(k) > 0$ — n–*th root of* $\mathrm{dn}(p)\,\mathrm{sn}(p) + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}(p) = 0$.

Bound of the cut time

$$\lambda \in C_1 \Rightarrow t^1_{\mathrm{MAX}} = \min(2p^1_z, 4K)\sigma,$$
$$\lambda \in C_2 \Rightarrow t^1_{\mathrm{MAX}} = 2Kk\sigma,$$
$$\lambda \in C_6 \Rightarrow t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|},$$
$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \Rightarrow t^1_{\mathrm{MAX}} = +\infty.$$

Theorem 1 (A. Ardentov, Yu. L. Sachkov)

For any $\lambda \in C$

$$t_{cut}(\lambda) \leq t^1_{\mathrm{MAX}}(\lambda)$$

Numerical solution of the optimal control problem: Reduction to the system of equations

$Y = \frac{y_t}{x_t}, Z = \frac{z_t}{x_t^2}, V = \frac{v_t}{x_t^3}$ are independent on α.

$$Y_1 = \frac{y_1}{x_1}, \quad z_1 = \frac{z_1}{x_1^2}, \quad V_1 = \frac{v_1}{x_1^3}.$$

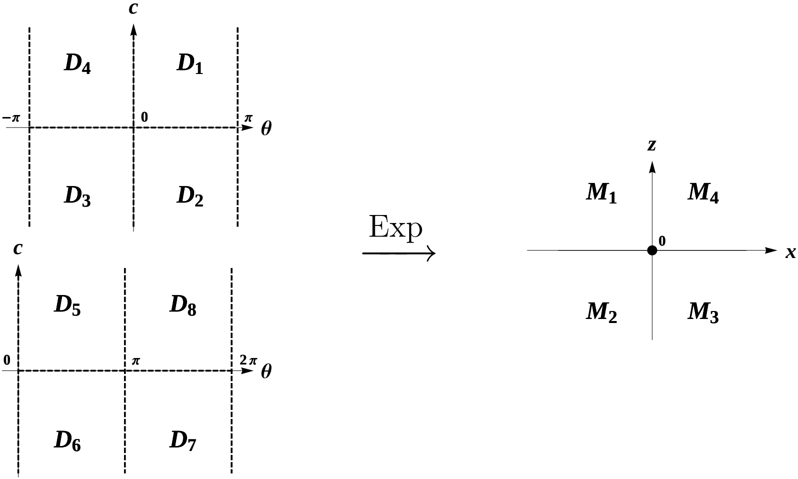
$$\begin{cases} Y(\tau, p, k) = Y_1, \\ Z(\tau, p, k) = Z_1, \\ V(\tau, p, k) = V_1. \end{cases}$$

Decomposition of the preimage of exponential mapping

$$
\begin{aligned}
&C = \cup_{i=1}^{8} D_i,\\
&D_1 \cap C_1 = \{\tau \in (0, K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_1 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_2 \cap C_1 = \{\tau \in (K, 2K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_2 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_3 \cap C_1 = \{\tau \in (2K, 3K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_3 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_4 \cap C_1 = \{\tau \in (3K, 4K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_4 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},
\end{aligned}
$$

where $p_{min}^1 = \min(p_z^1, 2K)$.

Correspondence between domains in image and preimage of exponential mapping



Conjecture: $\mathrm{Exp} : D_i \to M_i$ and $\mathrm{Exp} : D_{i+4} \to M_i$ are diffeomorphisms for $i \in \{1 \dots 4\}$.

Conjugate points

$$d_\nu \operatorname{Exp} \, : \, T_\nu N \to T_{q_t} M \text{ is degenerate,}$$

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)}(\nu) = 0.$$

$$t^1_{\text{conj}} = \min \{ t > 0 \mid t \text{ is a conjugate time along } \operatorname{Exp}(\lambda, s), \; s \geq 0 \} .$$

Theorem 2 (A. Ardentov, Yu. L. Sachkov)

For any $\lambda \in C$

$$t^1_{\text{MAX}}(\lambda) \leq t^1_{\text{conj}}(\lambda).$$

Results

- Nilpotent sub-Riemannian problem on the Engel group was considered.
- Extremal curves for this problem were found.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- Global upper bound of the cut time along extremal curves was proved.
- The estimate of the first conjugate time was proved.
- The description of global structure of the exponential map was get.
- Problem was reduced to solving of the system of three algebraic equations.
- Development of the software for numerical solution of the problem was started.

Plans

- Complete investigation of optimality of extremal curves and developing of the program for computing optimal curves for sub-Riemannian problem on the Engel group.
- Nilpotent approximation of nonholonomic systems in four-dimensional space with two-dimensional control (in particular, car with the trailer).