Extremal trajectories and Maxwell points in sub-Riemannian problem on the Engel group

A. A. Ardentov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia

aaa@pereslavl.ru

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Problem Statement

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix},$$

$$q = (x, y, z, v) \in \mathbb{R}^4, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$q(0) = q_0 = (0, 0, 0, 0)^T, \quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1)^T,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min \iff \int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min .$$

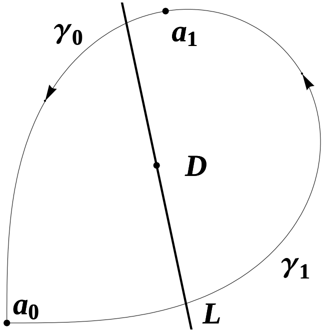
Geometric formulation of the problem

Given:

$a_0, a_1 \in \mathbb{R}^2$,
$\gamma_0 \subset \mathbb{R}^2$ connecting a_1 to a_0
$S \in \mathbb{R}$, line $L \subset \mathbb{R}^2$

Find:

$\gamma_1 \subset \mathbb{R}^2$ connecting a_0 to a_1,
s. t. $\gamma_1 \cup \gamma_0 = \partial D$,
$\text{area}(D) = S$,
center of mass of $D \in L$,
$\text{length}(\gamma_1) \to \min.$



Overview

- Parameterization of extremal curves.
- Symmetries of exponential mapping and construction of the Maxwell sets.
- Global bound of the cut time and necessary optimality conditions for extremal curves.
- Algorithm and software for numerical solution of the problem.

Known results for invariant sub-Riemannian problems on Lie groups

1. Three-dimensional Lie groups:
   - Heisenberg group (A. M. Vershik, V. Ya. Gershkovich 1986),
   - $SL(2), SO(3), S^3$ (U. Boscain, F. Rossi 2008),
   - $SE(2)$ (Yu. L. Sachkov 2010)
2. 5-dimensional nilpotent Lie group with growth vector $(2, 3, 5)$ (Yu.L.Sachkov 2006).
3. 6-dimensional nilpotent Lie group with growth vector $(3, 6)$ (O.M. Myasnichenko 2002).

Nilpotent sub-Riemannian problem on the Engel group

$$X_1 = (1, 0, -\frac{y}{2}, 0)^T, \qquad X_2 = (0, 1, \frac{x}{2}, \frac{x^2+y^2}{2})^T.$$

$$\begin{aligned}
&\mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, X_2, X_3, X_4),\\
&\dim \mathrm{Lie}(X_1, X_2)(q) = 4,\\
&[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,\\
&[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.
\end{aligned}$$

Growth vector (2, 3, 4).

Nilpotent approximation of nonholonomic control systems in four-dimensional space with two-dimensional control (e. g. car with trailer).

Controllability and existence of optimal curves

1. $X_1(q), \ldots, X_4(q)$ are linearly independent $\forall q \in \mathbb{R}^4 \xrightarrow{\text{Rashevskii–Chow theorem}}$ complete controllability.

2. Existence of optimal solutions is implied by Filippov theorem.

Pontryagin's maximum principle : Abnormal extremal trajectories

$$x = 0, \qquad y = \pm t, \qquad z = 0, \qquad v = \pm \frac{t^3}{6}.$$

Normal Hamiltonian system

$$
\begin{aligned}
\dot\theta &= c, && \theta \in S^1,\\
\dot c &= -\alpha \sin\theta, && c \in \mathbb{R},\\
\dot\alpha &= 0, && \alpha \in \mathbb{R},\\
\dot x &= -\sin\theta, &&\\
\dot y &= \cos\theta, &&\\
\dot z &= \frac{x\cos\theta + y\sin\theta}{2}, &&\\
\dot v &= \cos\theta\,\frac{x^2+y^2}{2}. &&
\end{aligned}
$$

$$
E = \frac{c^2}{2} - \alpha\cos\theta \in [-|\alpha|, +\infty).
$$

Equation of pendulum and physical meaning of α

$$\ddot{\theta} = -\alpha \sin\theta, \qquad \alpha = \frac{g}{L} = \text{const} \in \mathbb{R}$$

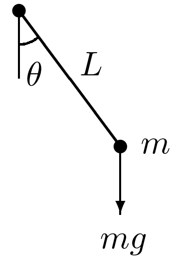


Figure: Mathematical pendulum with $\alpha > 0$

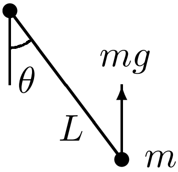


Figure: Mathematical pendulum with $\alpha < 0$

Stratification of phase cylinder of pendulum

$$C = T^*_{q_0}M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\}.$$

$$C = \cup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset,\ i \neq j.$$

$$C_i^+ = C_i \cap \{\alpha > 0\}, \qquad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\},$$

$$C_{i+}^{\pm} = C_i^{\pm} \cap \{c > 0\}, \qquad C_{i-}^{\pm} = C_i^{\pm} \cap \{c < 0\}, \qquad i \in \{2, 3\}.$$

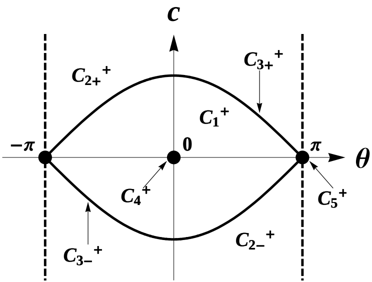


Figure: Stratification for $\alpha > 0$

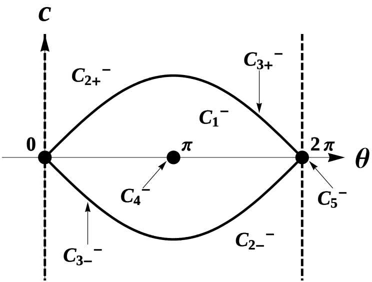


Figure: Stratification for $\alpha < 0$

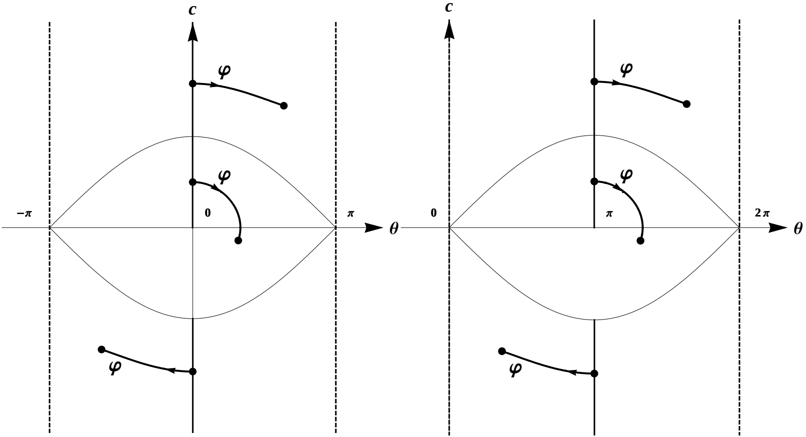
Elliptic coordinates in C^+

$$\lambda \in C_1^+,$$
$$k = \sqrt{\frac{E+\alpha}{2\alpha}} = \sqrt{\frac{c^2}{4\alpha} + \sin^2\frac{\theta}{2}} \in (0,1),$$
$$\sin\frac{\theta}{2} = k\,\mathrm{sn}(\sqrt{\alpha}\varphi), \qquad \cos\frac{\theta}{2} = \mathrm{dn}(\sqrt{\alpha}\varphi),$$
$$\frac{c}{2} = k\sqrt{\alpha}\,\mathrm{cn}(\sqrt{\alpha}\varphi), \qquad \varphi \in [0, 4K],$$

where $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}, \mathrm{E}$ are elliptic Jacobi's functions.

Equation of pendulum: $\dot{\varphi} = 1, \quad \dot{k} = \dot{\alpha} = 0.$

Elliptic coordinates (φ, k) in the phase cylinder of pendulum



Elliptic coordinates in C^-

Coordinates in the sets C_1^-, C_2^-, C_3^-:

$$\varphi(\theta, c, \alpha) = \varphi(\theta - \pi, c, -\alpha),$$
$$k(\theta, c, \alpha) = k(\theta - \pi, c, -\alpha).$$

Parametrization of extremal curves in the case $\alpha = 1$

$\lambda \in C_1^+$ (oscillations of pendulum) $\Rightarrow$

$$x_t = 2k(\operatorname{cn}\varphi_t - \operatorname{cn}\varphi),$$
$$y_t = 2(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) - t,$$
$$z_t = 2k(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi - \frac{y_t}{2}(\operatorname{cn}\varphi_t + \operatorname{cn}\varphi)),$$
$$v_t = \frac{y_t^3}{6} + 2k^2 \operatorname{cn}^2\varphi y_t - 4k^2 \operatorname{cn}\varphi(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi) +$$
$$+ 2k^2\left(\frac{2}{3}\operatorname{cn}\varphi_t \operatorname{dn}\varphi_t \operatorname{sn}\varphi_t - \frac{2}{3}\operatorname{cn}\varphi \operatorname{dn}\varphi \operatorname{sn}\varphi + \frac{1-k^2}{3k^2}t + \frac{2k^2-1}{3k^2}(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi))\right).$$

Symmetries of Hamiltonian system

Dilation of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta, \frac{c}{\sqrt{\alpha}}, 1, \sqrt{\alpha}x, \sqrt{\alpha}y, \alpha z, \alpha^{\frac{3}{2}}v, \sqrt{\alpha}t),$$

$$(\varphi, k, \alpha) \mapsto (\sqrt{\alpha}\varphi, k, 1).$$

Inversion of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t),$$

$$(\varphi, k, \alpha) \mapsto (\varphi, k, -\alpha).$$

Parametrization of extremal trajectories in general case with $\lambda \in \cup_{i=1}^{3} C_i$

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = (\frac{s_1}{\sigma} x_{\sigma t}, \frac{s_1}{\sigma} y_{\sigma t}, \frac{1}{\sigma^2} z_{\sigma t}, \frac{s_1}{\sigma^3} v_{\sigma t})(\sigma\varphi, k, 1),$$

where $\sigma = \sqrt{|\alpha|}, s_1 = \operatorname{sgn} \alpha.$

General case with $\alpha \neq 0$

$$
\begin{aligned}
&\lambda \in C_1 \Rightarrow \\
&x_t = \frac{2k\sigma}{\alpha}(\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)), \\
&y_t = \frac{2\sigma}{\alpha}(\mathrm{E}(\sigma\varphi_t) - \mathrm{E}(\sigma\varphi)) - \operatorname{sgn}\alpha t, \\
&z_t = \frac{2k}{|\alpha|}(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi) - \\
&\quad - \frac{\sigma k y_t}{2\alpha}(\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi))), \\
&v_t = \dots
\end{aligned}
$$

Parametrization of extremal curves for degenerate cases

$$\lambda \in C_4 \;\Rightarrow\; x_t = 0, \quad y_t = t \operatorname{sgn}\alpha, \quad z_t = 0, \quad v_t = \frac{t^3}{6} \operatorname{sgn}\alpha.$$

$$\lambda \in C_5 \;\Rightarrow\; x_t = 0, \quad y_t = -t \operatorname{sgn}\alpha, \quad z_t = 0, \quad v_t = -\frac{t^3}{6} \operatorname{sgn}\alpha.$$

$$\lambda \in C_6 \;\Rightarrow$$

$$x_t = \frac{\cos(ct+\theta) - \cos\theta}{c}, \quad y_t = \frac{\sin(ct+\theta) - \sin\theta}{c},$$

$$z_t = \frac{ct - \sin(ct)}{2c^2}, \quad v_t = -\frac{2c\cos\theta\, t - 4\sin(ct+\theta) + \sin(2ct+\theta)}{4c^3}.$$

$$\lambda \in C_7 \;\Rightarrow\; x_t = -t \sin\theta, \quad y_t = t \cos\theta, \quad z_t = 0, \quad v_t = \frac{\cos\theta}{6} t^3.$$

Euler elasticae

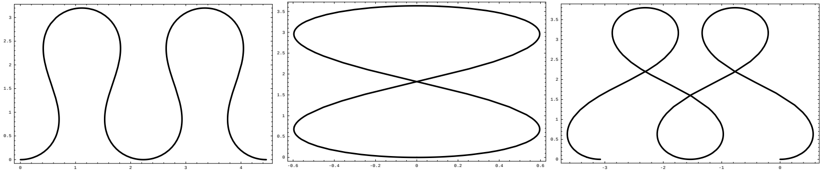


Figure: Inflectional elasticae

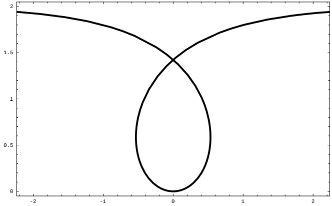


Figure: Critical elastica

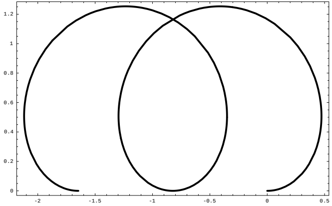


Figure: Non-inflectional elastica

Exponential mapping, Maxwell points and cut time

$$\mathrm{Exp} : C \times \mathbb{R}_+ \to M = \mathbb{R}^4,$$
$$\mathrm{Exp}(\lambda, t) = q_t,$$
$$\lambda = (\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+, \quad q_t \in M.$$

$$\mathrm{MAX} = \{(\lambda, t) \mid \exists \tilde{\lambda} \neq \lambda, \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(\tilde{\lambda}, t)\},$$

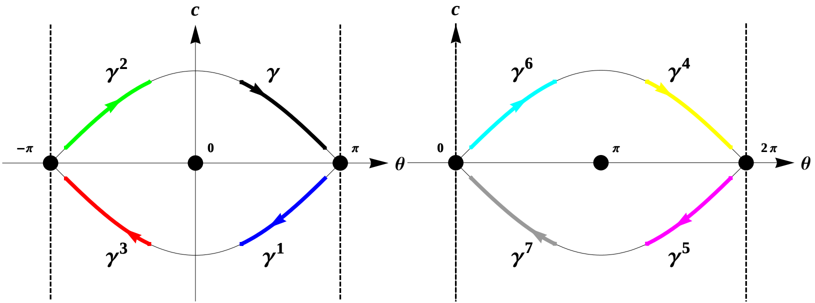
$t_{cut}(\lambda) = \sup\{t > 0 \mid \mathrm{Exp}(\lambda, s)$ is optimal for $s \in [0, t]\}$,
$t_{cut}(\lambda) \leq t$ for any $(\lambda, t) \in \mathrm{MAX}$.

Group of symmetries of exponential mapping

G	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
ε^1	Id	ε^3	ε^2	ε^5	ε^4	ε^7	ε^6
ε^2		Id	ε^1	ε^6	ε^7	ε^4	ε^5
ε^3			Id	ε^7	ε^6	ε^5	ε^4
ε^4				Id	ε^1	ε^2	ε^3
ε^5					Id	ε^3	ε^2
ε^6						Id	ε^1
ε^7							Id

Table: Multiplication in $G = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6, \varepsilon^7\}$

Reflections of trajectories of pendulum

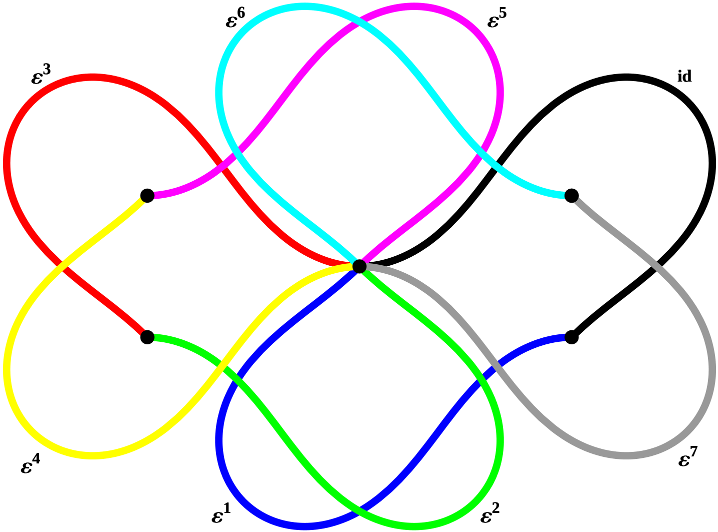


Reflections of trajectories of pendulum

$$\varepsilon^1 : \gamma \mapsto \gamma^1 = \{(\theta_s^1, c_s^1, \alpha^1)\} = \{(\theta_{t-s}, -c_{t-s}, \alpha) \mid s \in [0, t]\},$$
$$\varepsilon^2 : \gamma \mapsto \gamma^2 = \{(\theta_s^2, c_s^2, \alpha^2)\} = \{(-\theta_{t-s}, c_{t-s}, \alpha) \mid s \in [0, t]\},$$
$$\varepsilon^3 : \gamma \mapsto \gamma^3 = \{(\theta_s^3, c_s^3, \alpha^3)\} = \{(-\theta_s, -c_s, \alpha) \mid s \in [0, t]\},$$
$$\varepsilon^4 : \gamma \mapsto \gamma^4 = \{(\theta_s^4, c_s^4, \alpha^4)\} = \{(\theta_s + \pi, c_s, -\alpha) \mid s \in [0, t]\},$$
$$\varepsilon^5 : \gamma \mapsto \gamma^5 = \{(\theta_s^5, c_s^5, \alpha^5)\} = \{(\theta_{t-s} + \pi, -c_{t-s}, -\alpha) \mid s \in [0, t]\},$$
$$\varepsilon^6 : \gamma \mapsto \gamma^6 = \{(\theta_s^6, c_s^6, \alpha^6)\} = \{(-\theta_{t-s} + \pi, c_{t-s}, -\alpha) \mid s \in [0, t]\},$$
$$\varepsilon^7 : \gamma \mapsto \gamma^7 = \{(\theta_s^7, c_s^7, \alpha^7)\} = \{(-\theta_s + \pi, -c_s, -\alpha) \mid s \in [0, t]\},$$

where $\gamma = \{(\theta_s, c_s, \alpha) \mid s \in [0, t]\}$.

Reflections of Euler elasticae



Action of ε^i in the preimage of exponential mapping

$$\varepsilon^i : C \times \mathbb{R} \to C \times \mathbb{R}, \qquad \varepsilon^i(\theta, c, \alpha, t) = (\theta^i, c^i, \alpha^i, t),$$

$$(\theta^1, c^1, \alpha^1) = (\theta_t, -c_t, \alpha), \qquad (\theta^2, c^2, \alpha^2) = (-\theta_t, c_t, \alpha),$$

$$(\theta^3, c^3, \alpha^3) = (-\theta_t, -c_t, \alpha), \qquad (\theta^4, c^4, \alpha^4) = (\theta_t + \pi, c_t, -\alpha),$$

$$(\theta^5, c^5, \alpha^5) = (\theta_t + \pi, -c_t, -\alpha), \qquad (\theta^6, c^6, \alpha^6) = (-\theta_t + \pi, c_t, -\alpha),$$

$$(\theta^7, c^7, \alpha^7) = (-\theta_t + \pi, -c_t, -\alpha).$$

Action of ε^i in the image of exponential mapping

$$\varepsilon^i : M \to M, \quad \varepsilon^i(q) = \varepsilon^i(x, y, z, v) = q^i = (x^i, y^i, z^i, v^i),$$
$$(x^1, y^1, z^1, v^1) = (x, y, -z, v - xz),$$
$$(x^2, y^2, z^2, v^2) = (-x, y, z, v - xz),$$
$$(x^3, y^3, z^3, v^3) = (-x, y, -z, v),$$
$$(x^4, y^4, z^4, v^4) = (-x, y, -z, -v),$$
$$(x^5, y^5, z^5, v^5) = (-x, -y, -z, -v + xz),$$
$$(x^6, y^6, z^6, v^6) = (x, -y, z, -v + xz),$$
$$(x^7, y^7, z^7, v^7) = (x, -y, -z, -v).$$

Reflections as symmetries of Exp

Proposition

Reflection ε^i is a symmetry of exponential mapping for any $i = 1, \dots, 7$, i. e.,

$$\varepsilon^i \circ \mathrm{Exp}(\theta, c, \alpha, t) = \mathrm{Exp} \circ \varepsilon^i(\theta, c, \alpha, t),$$
$$(\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+.$$

$$\mathrm{MAX}^i = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t)\},$$
$$\lambda = (\theta, c, \alpha), \quad \lambda^i = (\theta^i, c^i, \alpha^i) = \varepsilon^i(\lambda).$$

Fixed points of ε^i in the image of exponential mapping

$$\mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t) \iff \varepsilon^i(q_t) = q_t.$$

Lemma

1. $\varepsilon^1(q) = q \iff z = 0,$
2. $\varepsilon^2(q) = q \iff x = 0,$
3. $\varepsilon^3(q) = q \iff x^2 + z^2 = 0,$
4. $\varepsilon^4(q) = q \iff x^2 + y^2 + v^2 = 0,$
5. $\varepsilon^5(q) = q \iff x^2 + y^2 + z^2 + v^2 = 0,$
6. $\varepsilon^6(q) = q \iff y^2 + (2v - xz)^2 = 0,$
7. $\varepsilon^7(q) = q \iff y^2 + z^2 + v^2 = 0.$

Fixed points of ε^i in the preimage of exponential mapping

Proposition

If $(\lambda, t) \in C \times \mathbb{R}_+$, $\varepsilon^i(\lambda, t) = (\lambda^i, t)$ *then:*

1. $\lambda^1 = \lambda \iff \begin{cases} \operatorname{cn}\tau = 0 \text{ if } \lambda \in C_1 \\ \text{is impossible if } \lambda \in C_2 \cup C_3 \cup C_6 \end{cases}$
2. $\lambda^2 = \lambda \iff \begin{cases} \operatorname{sn}\tau = 0 \text{ if } \lambda \in C_1 \\ \operatorname{sn}\tau \operatorname{cn}\tau = 0 \text{ if } \lambda \in C_2 \\ \tau = 0 \text{ if } \lambda \in C_3 \\ 2\theta + ct = 2\pi n \text{ if } \lambda \in C_6 \end{cases}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2k}.$$

Complete description of the Maxwell sets for $\varepsilon^1, \varepsilon^2$

Theorem

1. $\mathrm{MAX}^1 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = p_z^n(k), n \in \mathbb{N}, \mathrm{cn}(\tau) \neq 0\}$,
2. $\mathrm{MAX}^1 \cap N_2 = \mathrm{MAX}^1 \cap N_3 = \mathrm{MAX}^1 \cap N_6 = \emptyset$,
3. $\mathrm{MAX}^2 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = 2Kn, n \in \mathbb{N}, \mathrm{sn}(\tau) \neq 0\}$,
4. $\mathrm{MAX}^2 \cap N_2 = \{(\lambda, t) \in N_2 \mid p = Kn, n \in \mathbb{N}, \mathrm{sn}(\tau)\,\mathrm{cn}(\tau) \neq 0\}$,
5. $\mathrm{MAX}^2 \cap N_3 = \emptyset$,
6. $\mathrm{MAX}^2 \cap N_6 = \{(\lambda, t) \in N_6 \mid tc = 2\pi n, \theta \neq \pi k, n, k \in \mathbb{Z}\}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2},$$
$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2k}.$$

$p_z^n(k) > 0$ — *n–th root of* $\mathrm{dn}(p)\,\mathrm{sn}(p) + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}(p) = 0$.

Bound of the cut time

$$\lambda \in C_1 \Rightarrow \mathbf{t} = \min(2p_z^1, 4K)\sigma,$$
$$\lambda \in C_2 \Rightarrow \mathbf{t} = 2Kk\sigma,$$
$$\lambda \in C_6 \Rightarrow \mathbf{t} = \frac{2\pi}{|c|},$$
$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \Rightarrow \mathbf{t} = +\infty.$$

Theorem (A. A., Yu. Sachkov)

For any $\lambda \in C$

$$t_{cut}(\lambda) \leq \mathbf{t}(\lambda)$$

Numerical solution of the optimal control problem: Reduction to the system of equations

$Y = \frac{y_t}{x_t}, Z = \frac{z_t}{x_t^2}, V = \frac{v_t}{x_t^3}$ are independent on α.

$$Y_1 = \frac{y_1}{x_1}, \quad z_1 = \frac{z_1}{x_1^2}, \quad V_1 = \frac{v_1}{x_1^3}.$$

$$\begin{cases} Y(\tau, p, k) = Y_1, \\ Z(\tau, p, k) = Z_1, \\ V(\tau, p, k) = V_1. \end{cases}$$

Decomposition of the preimage of exponential mapping

$$
\begin{aligned}
&C = \cup_{i=1}^4 D_i,\\
&D_1 \cap C_1 = \{\tau \in (0, K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_1 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_2 \cap C_1 = \{\tau \in (K, 2K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_2 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_3 \cap C_1 = \{\tau \in (2K, 3K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_3 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},\\
&D_4 \cap C_1 = \{\tau \in (3K, 4K), p \in (0, p_{min}^1), k \in (0,1)\},\\
&D_4 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0,1), \operatorname{sgn} c = 1\},
\end{aligned}
$$

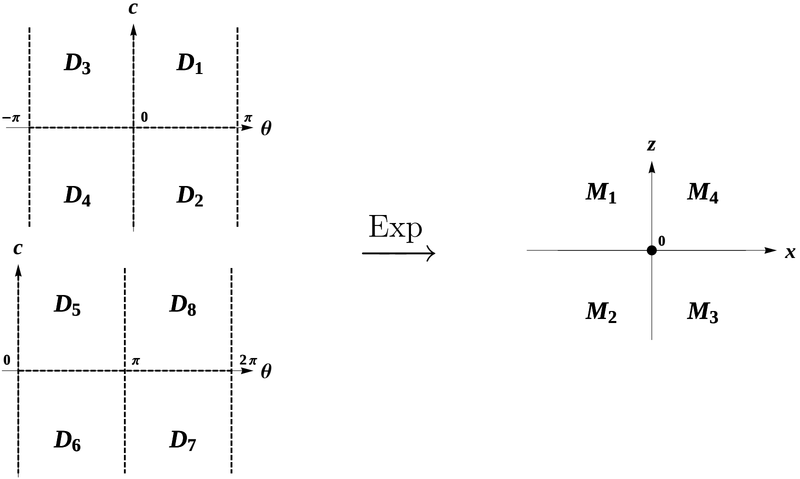
where $p_{min}^1 = \min(p_z^1, 2K)$.

Correspondence between domains in image and preimage of exponential mapping

$$M = \cup_{i=1}^{4} M_i,$$
$$M_1 = \{(x, y, z, v) \in \mathbb{R}^4 \mid x > 0, z > 0\},$$
$$M_2 = \{(x, y, z, v) \in \mathbb{R}^4 \mid x < 0, z < 0\},$$
$$M_3 = \{(x, y, z, v) \in \mathbb{R}^4 \mid x > 0, z < 0\},$$
$$M_4 = \{(x, y, z, v) \in \mathbb{R}^4 \mid x < 0, z > 0\}.$$

$$q_1 \in M_1 \Rightarrow (\tau, p, k) \in D_1 \cup D_5,$$
$$q_1 \in M_2 \Rightarrow (\tau, p, k) \in D_2 \cup D_6,$$
$$q_1 \in M_3 \Rightarrow (\tau, p, k) \in D_3 \cup D_7,$$
$$q_1 \in M_4 \Rightarrow (\tau, p, k) \in D_4 \cup D_8.$$

Correspondence between domains in image and preimage of exponential mapping



Conjecture: $\mathrm{Exp} : D_i \to M_i$ and $\mathrm{Exp} : D_{i+4} \to M_i$ are diffeomorphisms.

Results

- Nilpotent sub-Riemannian problem on the Engel group was considered.
- Extremal curves for this problem were found.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- Global upper bound of the cut time along extremal curves was proved.
- Problem was reduced to solving of the system of three algebraic equations.
- Development of the software for numerical solution of the problem was started.

Plans

- Complete investigation of optimality of extremal curves and developing of the program for computing optimal curves for sub-Riemannian problem on the Engel group.
- Nilpotent approximation of nonholonomic systems in four-dimensional space with two-dimensional control (in particular, car with the trailer).