# Nilpotent sub-Riemannian problem on the Engel group

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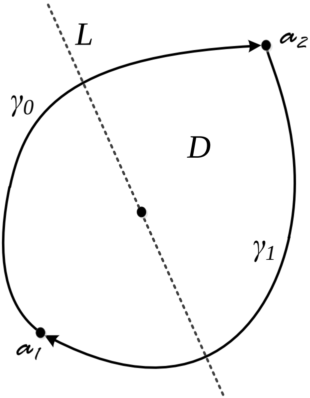
# Geometric formulation of the problem

**Given:**
$a_1, a_2 \in \mathbb{R}^2$,
$\gamma_0 \subset \mathbb{R}^2$ connecting $a_1$ to $a_2$,
$S \in \mathbb{R}$, line $L \subset \mathbb{R}^2$.

**Find:**
$\gamma_1 \subset \mathbb{R}^2$ connecting $a_2$ to $a_1$,
s. t. $\gamma_1 \cup \gamma_0 = \partial D$,
$\mathrm{area}(D) = S$,
center of mass of $D \in L$,
$\mathrm{length}(\gamma_1) \to \min$.



# Left-invariant sub-Riemannian problem

$$q = (x, y, z, v)^T \in M = \mathbf{R}^4, \quad (u_1, u_2) \in \mathbf{R}^2,$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix},$$

$$\dot{q} = u_1 X_1 + u_2 X_2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min \quad \Leftrightarrow \quad \int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min\,.$$

# Known results for invariant sub-Riemannian problems on Lie groups

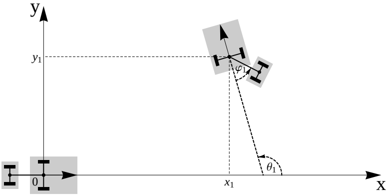
- Three-dimensional Lie groups
  - Heisenberg group (A.M. Vershik, V.Ya. Gershkovich 1986);
  - $\mathrm{SL}(2)$, $\mathrm{SO}(3)$, $S^3$ (U. Boscain, F. Rossi 2008);
  - $\mathrm{SE}(2)$ (Yu.L. Sachkov 2010).
- Classification of all 4-dimensional sub-Riemannian homogeneous spaces of Engel Type (Almeida, 2013).
- 5-dimensional nilpotent Lie group with growth vector (2, 3, 5) (Yu.L. Sachkov 2006).
- 6-dimensional nilpotent Lie group with growth vector (3, 6) (O.M. Myasnichenko 2002).

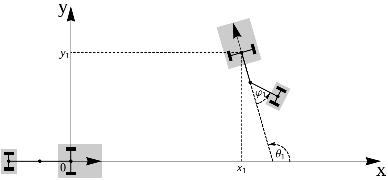
# Outline

- Parameterization of normal geodesic.
- Symmetries of exponential mapping and construction of the Maxwell sets.
- Description of the first Maxwell time, equivalence with the cut time.
- Parameterization of optimal solutions.
- Studying of cut locus.

# Nilpotent approximation of nonholonomic control systems

$$q = (x, y, \theta, \varphi)^T \in M = \mathbb{R}^2_{x,y} \times S^1_\theta \times S^1_\varphi,$$
$$\dot{q} = u_1 X_1 + u_2 X_2.$$



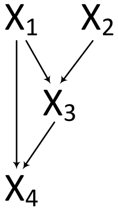




$$X_1 = (\cos\theta, \sin\theta, 0, -\sin\varphi)^T,$$
$$X_2 = (0, 0, 1, 1)^T.$$

$$X_1 = (\cos\theta, \sin\theta, 0, -\sin\varphi)^T,$$
$$X_2 = (0, 0, 1, -1-\cos\varphi)^T.$$

# Nilpotent Lie algebra.
# Controllability and existence of optimal curves

$$X_3 = [X_1, X_2] = (0, 0, 1, x)^T,$$
$$X_4 = [X_1, [X_1, X_2]] = (0, 0, 0, 1)^T,$$
$$[X_2, [X_1, X_2]] = 0,$$
$$[X_1, [X_1, [X_1, X_2]]] = 0,$$
$$[X_2, [X_1, [X_1, X_2]]] = 0.$$




Growth vector (2, 3, 4).

$$\mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, X_2, X_3, X_4),$$
$$\dim \mathrm{Lie}(X_1, X_2)(q) = 4 \quad \Longrightarrow \quad \text{complete controllability.}$$

Existence of optimal solutions is implied by Filippov's theorem.

# Pontryagin's maximum principle:

$$\psi_0 \le 0, \qquad \psi = (\psi_1, \psi_2, \psi_3, \psi_4),$$

$$H(\psi_0, \psi, q, u) = \psi_0 \frac{u_1^2 + u_2^2}{2} + \langle \psi, u_1 X_1 + u_2 X_2 \rangle =$$

$$= \psi_0 \frac{u_1^2 + u_2^2}{2} + \psi_1 u_1 + \psi_2 u_2 + \psi_3 \frac{x u_2 - y u_1}{2} + \psi_4 \frac{x^2 + y^2}{2} u_2.$$

$$\begin{cases} \dot{\psi} = -\frac{\partial H}{\partial q}, \\ \dot{q} = \frac{\partial H}{\partial \psi}, \end{cases}$$

$$\max_{u \in \mathbb{R}^2} H(\psi(t), \hat{q}(t), u) = H(\psi(t), \hat{q}(t), \hat{u}(t)).$$

# Pontryagin's maximum principle: Coordinates for the normal case ($\psi_0 = -1$)

$$h_i = \langle \psi, X_i \rangle \quad \Longrightarrow \quad H = \frac{1}{2}(h_1^2 + h_2^2) = \frac{1}{2}.$$

$$\begin{aligned} h_1 &= \cos(\theta + \frac{\pi}{2}), \\ h_2 &= \sin(\theta + \frac{\pi}{2}), \\ h_3 &= c, \\ h_4 &= \alpha. \end{aligned}$$

# Pontryagin's maximum principle: Normal Hamiltonian system and symmetries

$$
\begin{aligned}
\dot\theta &= c, & \theta &\in S^1,\\
\dot c &= -\alpha \sin\theta, & c &\in \mathbb{R},\\
\dot\alpha &= 0, & \alpha &\in \mathbb{R},\\
\dot x &= -\sin\theta, & &\\
\dot y &= \cos\theta, & &\\
\dot z &= \frac{x\cos\theta + y\sin\theta}{2}, & &\\
\dot v &= \cos\theta\,\frac{x^2+y^2}{2}. & &
\end{aligned}
$$

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta, \frac{c}{M}, \frac{\alpha}{M^2}, Mx, My, M^2 z, M^3 v, Mt),\ M > 0.$$

$$(\theta, c, \alpha < 0, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha > 0, -x, -y, z, -v, t).$$

# Equation of pendulum and physical meaning of $\alpha$

$$\ddot{\theta} = -\alpha \sin\theta, \qquad \alpha = \frac{g}{L} = \text{const} \in \mathbb{R}$$

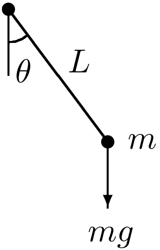



Figure : Pendulum with $\alpha > 0$

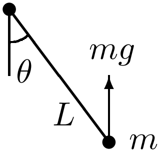



Figure : Pendulum with $\alpha < 0$

$$E = \frac{c^2}{2} - \alpha\cos\theta \in [-|\alpha|, +\infty).$$

# Stratification of phase cylinder of pendulum

$$C = T^*_{q_0}M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\}.$$
$$C = \cup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset,\ i \neq j.$$

$$C_i^+ = C_i \cap \{\alpha > 0\}, \qquad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\},$$
$$C_{i+}^\pm = C_i^\pm \cap \{c > 0\}, \qquad C_{i-}^\pm = C_i^\pm \cap \{c < 0\}, \qquad i \in \{2, 3\}.$$

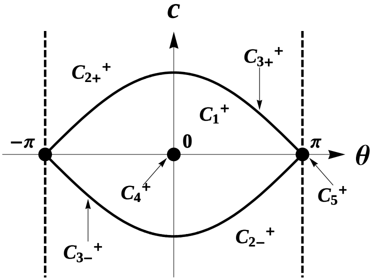


Figure : Stratification for $\alpha > 0$

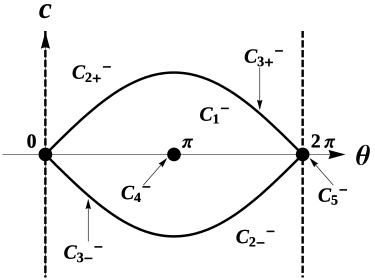
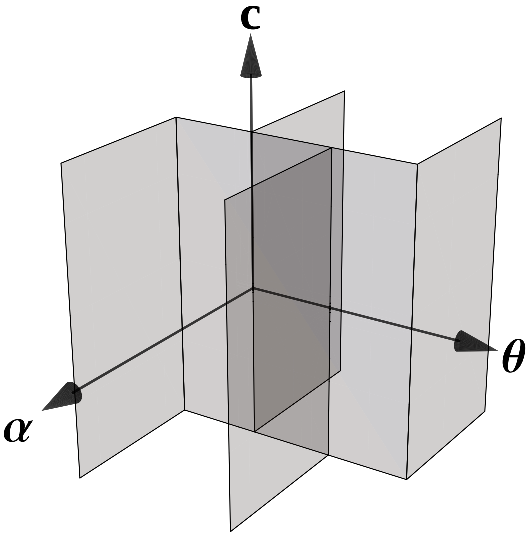
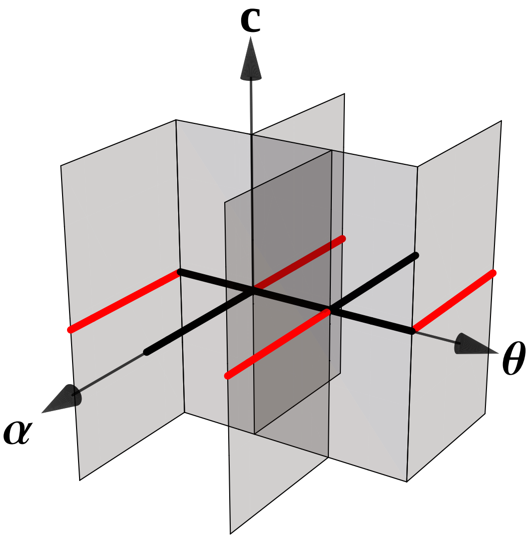


Figure : Stratification for $\alpha < 0$

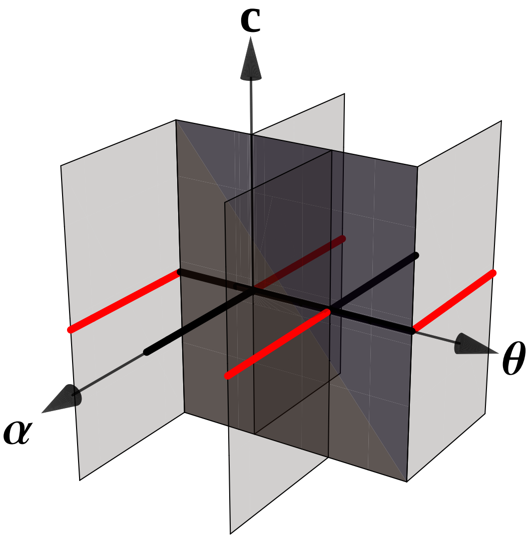
# Phase cylinder



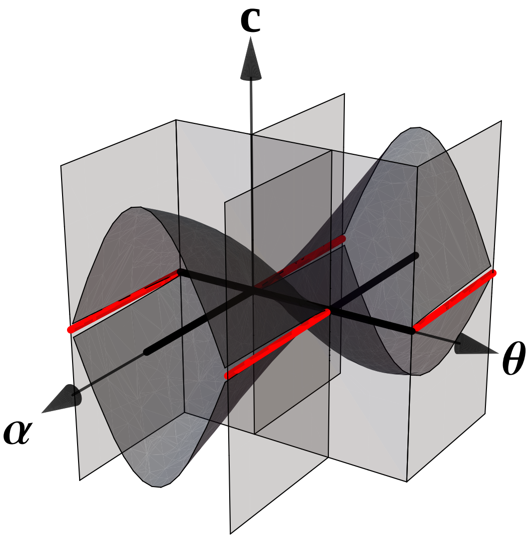
# Projection on XY: Lines



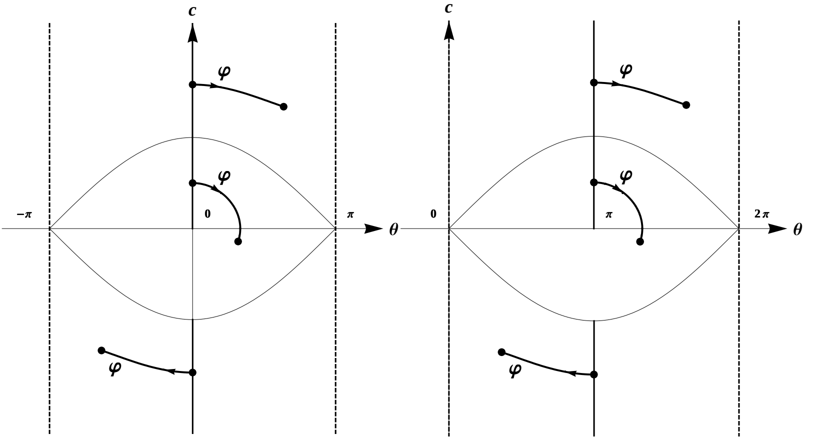
# Projection on XY: Circles



# Critical type



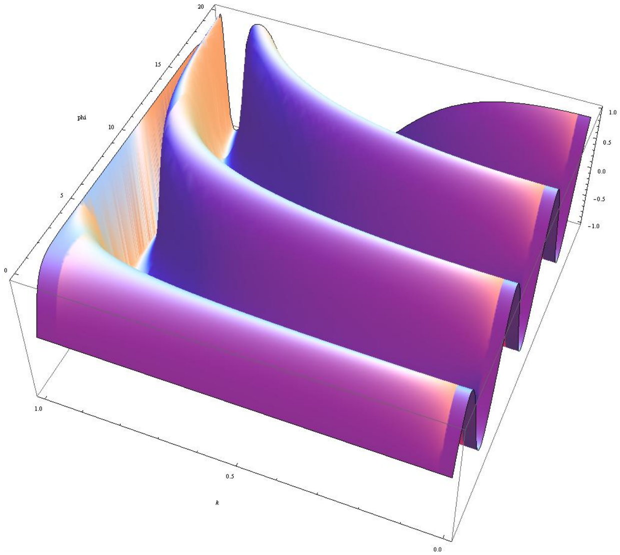
# Elliptic coordinates $(\varphi, k)$ in the phase cylinder of pendulum




Equation of pendulum: $\dot{\varphi} = 1, \quad \dot{k} = \dot{\alpha} = 0.$

$\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ are elliptic Jacobi functions; $\mathrm{F}, \mathrm{E}$ — elliptic integrals.

# Elliptic Jacobi functions: $\mathrm{sn}(\varphi) = \mathrm{sn}(\varphi, k)$



# Parameterization of extremal curves

oscillations of pendulum:

$$x_t = \frac{2k\sigma}{\alpha}\Big(\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)\Big), \qquad y_t = \frac{2\sigma}{\alpha}\Big(\operatorname{E}(\sigma\varphi_t) - \operatorname{E}(\sigma\varphi)\Big) - \operatorname{sgn}\alpha\; t,$$

$$z_t = \frac{2k}{|\alpha|}\Big(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi) - \frac{\sigma k y_t}{2\alpha}\Big(\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi)\Big)\Big),$$

$$v_t = \frac{2k^2}{\sigma\alpha}\Big(\frac{2}{3}\operatorname{cn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t)\operatorname{sn}(\sigma\varphi_t) - \frac{2}{3}\operatorname{cn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi)\operatorname{sn}(\sigma\varphi) + \frac{1-k^2}{3k^2}\sigma t + \frac{2k^2-1}{3k^2}\Big(\operatorname{E}(\sigma\varphi_t) -$$

$$- \operatorname{E}(\sigma\varphi)\Big)\Big) + \frac{y_t^3}{6} + \frac{2k^2}{|\alpha|}\operatorname{cn}^2(\sigma\varphi)y_t - \frac{4k^2}{\sigma\alpha}\operatorname{cn}(\sigma\varphi)\Big(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi)\Big).$$

rotations of pendulum:

$$x_t = \frac{2\sigma\operatorname{sgn} c}{\alpha k}\Big(\operatorname{dn}(\sigma\psi_t) - \operatorname{dn}(\sigma\psi)\Big), \qquad y_t = \frac{k^2-2}{k^2}\operatorname{sgn}\alpha\, t + \frac{2\sigma}{\alpha k}\Big(\operatorname{E}(\sigma\psi_t) - \operatorname{E}(\sigma\psi)\Big),$$

$$z_t = -\frac{x_t y_t}{2} - \frac{2\sigma\operatorname{sgn} c\operatorname{dn}(\sigma\psi)}{\alpha k}y_t + \frac{2\operatorname{sgn} c}{|\alpha|}\Big(\operatorname{cn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \operatorname{cn}(\sigma\psi)\operatorname{sn}(\sigma\psi)\Big),$$

$$v_t = \frac{4}{\sigma\alpha k}\Big(\frac{1}{3}\operatorname{cn}(\sigma\psi_t)\operatorname{dn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \frac{1}{3}\operatorname{cn}(\sigma\psi)\operatorname{dn}(\sigma\psi)\operatorname{sn}(\sigma\psi) - \frac{1-k^2}{3k^3}\sigma t - \frac{k^2-2}{6k^2}\Big(\operatorname{E}(\sigma\psi_t) -$$

$$- \operatorname{E}(\sigma\psi)\Big)\Big) - \frac{4}{\sigma\alpha k}\operatorname{dn}(\sigma\psi)\big(\operatorname{cn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \operatorname{cn}(\sigma\psi)\operatorname{sn}(\sigma\psi)\big) + \frac{y_t^3}{6} + \frac{2y_t}{|\alpha|k^2}\operatorname{dn}^2(\sigma\psi).$$

# Euler elasticae

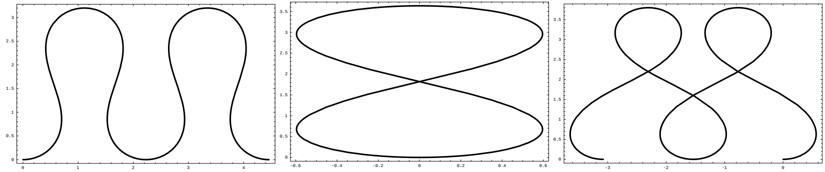



Figure : Inflectional elasticae

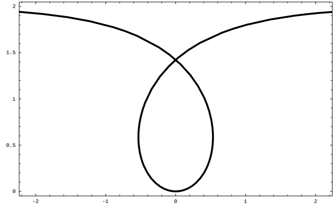



Figure : Critical elastica

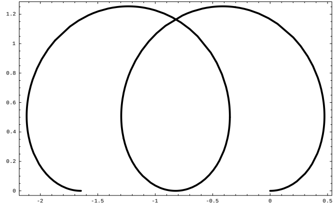



Figure : Non-inflectional elastica

# Exponential mapping, Maxwell points, conjugate and cut time

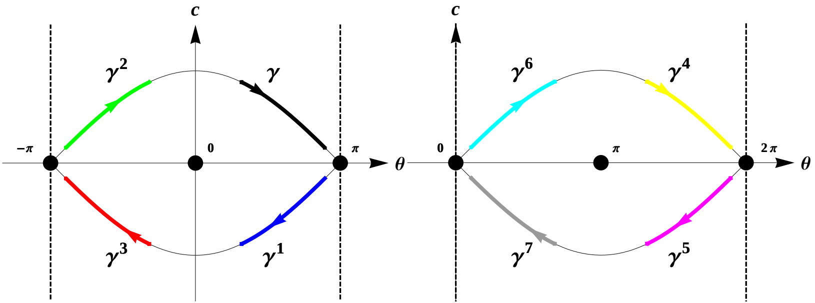
$$\text{Exp} : C \times \mathbb{R}_+ \to M = \mathbb{R}^4,$$
$$\text{Exp}(\lambda, t) = q_t,$$
$$\lambda = (\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+, \quad q_t \in M.$$

$$t_{cut}(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is globally optimal for } s \in [0, t]\},$$
$$t^1_{conj}(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is locally optimal for } s \in [0, t]\},$$

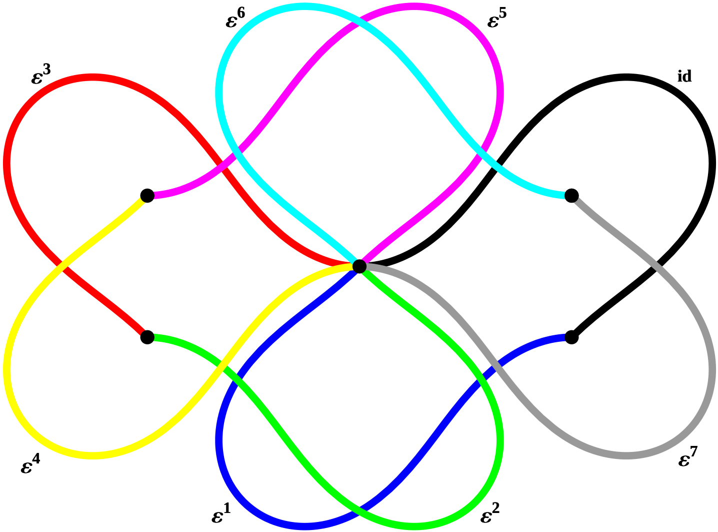
$$\text{MAX} = \{(\lambda, t_{\text{MAX}}) \mid \exists \tilde{\lambda} \neq \lambda, \text{Exp}(\lambda, t_{\text{MAX}}) = \text{Exp}(\tilde{\lambda}, t_{\text{MAX}})\},$$

$$t_{cut}(\lambda) = \min(t_{\text{MAX}}(\lambda), t^1_{conj}(\lambda)), \qquad \forall \lambda \in C$$
$$\Rightarrow \qquad t_{cut}(\lambda) \leq t_{\text{MAX}}(\lambda), \quad t_{cut}(\lambda) \leq t^1_{conj}(\lambda).$$

# Reflections of trajectories of pendulum



# Reflections of Euler elasticae



# Reflections as symmetries of $\mathrm{Exp}$

Proposition

*Reflection $\varepsilon^i$ is a symmetry of exponential mapping for any $i = 1, \dots, 7$, i. e.,*

$$\varepsilon^i \circ \mathrm{Exp}(\theta, c, \alpha, t) = \mathrm{Exp} \circ \varepsilon^i(\theta, c, \alpha, t),$$
$$(\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+.$$

$$\mathrm{MAX}^i = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t)\},$$
$$\lambda = (\theta, c, \alpha), \quad \lambda^i = (\theta^i, c^i, \alpha^i) = \varepsilon^i(\lambda).$$

# Fixed points of $\varepsilon^i$ in the image of exponential mapping

$$\mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t) \iff \varepsilon^i(q_t) = q_t.$$

Lemma

1. $\varepsilon^1(q) = q \iff z = 0$,
2. $\varepsilon^2(q) = q \iff x = 0$,
3. $\varepsilon^3(q) = q \iff x^2 + z^2 = 0$,
4. $\varepsilon^4(q) = q \iff x^2 + y^2 + v^2 = 0$,
5. $\varepsilon^5(q) = q \iff x^2 + y^2 + z^2 + v^2 = 0$,
6. $\varepsilon^6(q) = q \iff y^2 + (2v - xz)^2 = 0$,
7. $\varepsilon^7(q) = q \iff y^2 + z^2 + v^2 = 0$.

# Fixed points of $\varepsilon^i$ in the preimage of exponential mapping

Proposition

*If* $(\lambda, t) \in C \times \mathbb{R}_+$, $\varepsilon^i(\lambda, t) = (\lambda^i, t)$ *then:*

1. $\lambda^1 = \lambda \iff \begin{cases} \operatorname{cn}\tau = 0 \text{ if } \lambda \in C_1 \\ \text{is impossible if } \lambda \in C_2 \cup C_3 \cup C_6 \end{cases}$
2. $\lambda^2 = \lambda \iff \begin{cases} \operatorname{sn}\tau = 0 \text{ if } \lambda \in C_1 \\ \operatorname{sn}\tau \operatorname{cn}\tau = 0 \text{ if } \lambda \in C_2 \\ \tau = 0 \text{ if } \lambda \in C_3 \\ 2\theta + ct = 2\pi n \text{ if } \lambda \in C_6 \end{cases}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \qquad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \qquad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2k}.$$

# Bound of the cut time

$$
\begin{aligned}
\lambda \in C_1 \quad & \Rightarrow \quad t^1_{\mathrm{MAX}} = \min(2p^1_z, 4K)\sigma, \\
\lambda \in C_2 \quad & \Rightarrow \quad t^1_{\mathrm{MAX}} = 2Kk\sigma, \\
\lambda \in C_6 \quad & \Rightarrow \quad t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|}, \\
\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \quad & \Rightarrow \quad t^1_{\mathrm{MAX}} = +\infty,
\end{aligned}
$$

$p^1_z > 0$ — 1-st root of $\mathrm{dn}(p)\,\mathrm{sn}(p) + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}(p) = 0.$

Theorem (A. A., Yu. Sachkov)

*For any* $\lambda \in C$

$$t_{cut}(\lambda) \leq t^1_{\mathrm{MAX}}(\lambda).$$

# Decomposition of the preimage and the image of exponential mapping

$$L_1 = \{(\lambda, t) \in C \times \mathbb{R}_+ | \theta_{\frac{t}{2}}(\lambda) \in (0, \pi), c_{\frac{t}{2}}(\lambda) > 0\},$$
$$L_2 = \{(\lambda, t) \in C \times \mathbb{R}_+ | \theta_{\frac{t}{2}}(\lambda) \in (0, \pi), c_{\frac{t}{2}}(\lambda) < 0\},$$
$$L_3 = \{(\lambda, t) \in C \times \mathbb{R}_+ | \theta_{\frac{t}{2}}(\lambda) \in (-\pi, 0), c_{\frac{t}{2}}(\lambda) < 0\},$$
$$L_4 = \{(\lambda, t) \in C \times \mathbb{R}_+ | \theta_{\frac{t}{2}}(\lambda) \in (-\pi, 0), c_{\frac{t}{2}}(\lambda) > 0\}.$$

$$M_1 = \{q = (x, y, z, v) \in \mathbb{R}^4 | x < 0, z > 0\},$$
$$M_2 = \{q = (x, y, z, v) \in \mathbb{R}^4 | x < 0, z < 0\},$$
$$M_3 = \{q = (x, y, z, v) \in \mathbb{R}^4 | x > 0, z < 0\},$$
$$M_4 = \{q = (x, y, z, v) \in \mathbb{R}^4 | x > 0, z > 0\}.$$

# The first conjugate time

Theorem (A. A., Yu. Sachkov)
*For any* $\lambda \in C^1$

$$t^1_{\mathrm{MAX}}(\lambda) \leq t^1_{conj}(\lambda).$$

[1] A. A., Yu. L. Sachkov, Conjugate points in nilpotent sub-Riemannian problem on the Engel group, Journal of Mathematical Sciences, Vol. 195, No. 3, December, 2013.

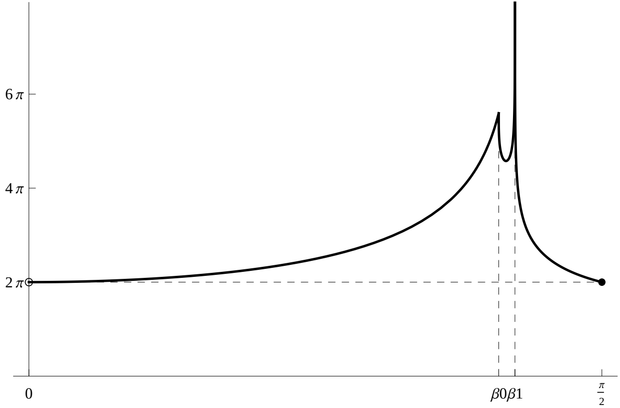
# Hadamard theorem on global diffeomorphism

$L_i, M_i$ are smooth manifolds.
$\mathrm{Exp} : L_i \to M_i$ is smooth mapping.

1. $\dim M_i = \dim L_i = 4$.
2. $L_i$ is connected, $M_i$ is connected.
3. $M_i$ is simply connected.
4. $\mathrm{Exp} : L_i \to M_i$ is non-degenerate.
5. $\mathrm{Exp} : L_i \to M_i$ is proper (preimage of compact is compact).

$$\Longrightarrow \quad \mathrm{Exp} : L_i \to M_i \text{ is diffeomorphism.}$$

# Cut time

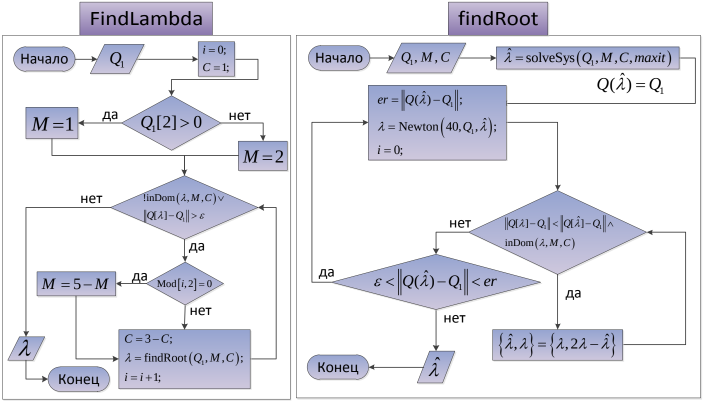



Theorem (A. A., Yu. Sachkov)

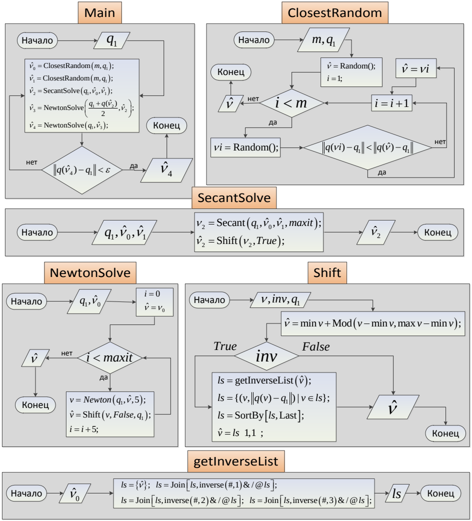
*For any* $\lambda \in C$

$$t_{cut}(\lambda) = t^1_{\mathrm{MAX}}(\lambda).$$

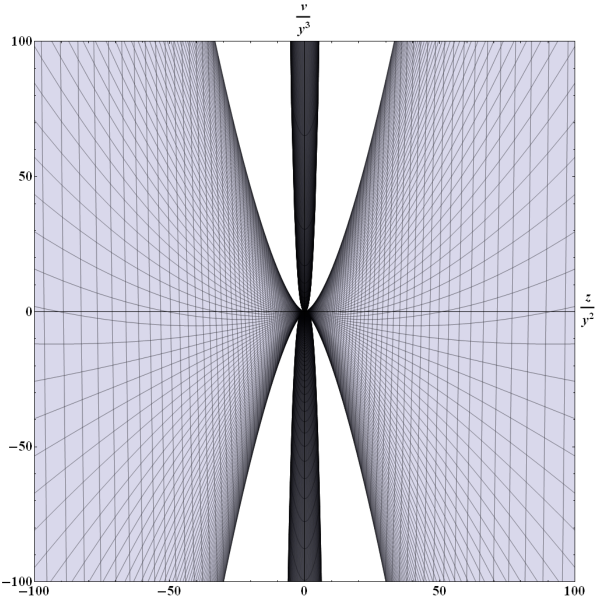
# Scheme of the algorithm for finding an optimal solution



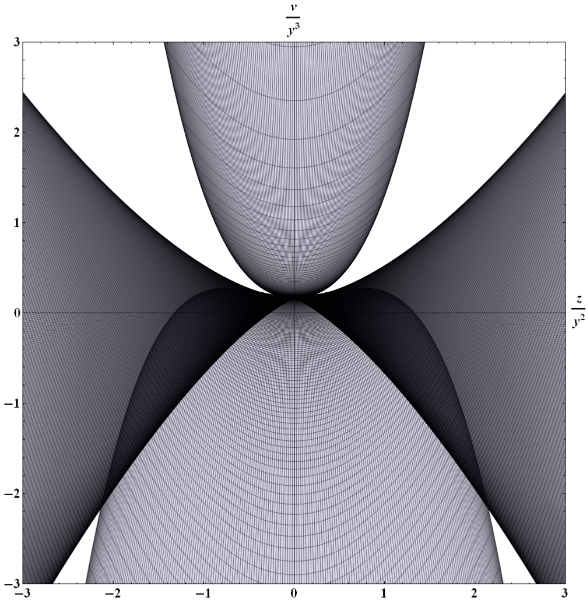
# Scheme of the algorithm for solving the system function



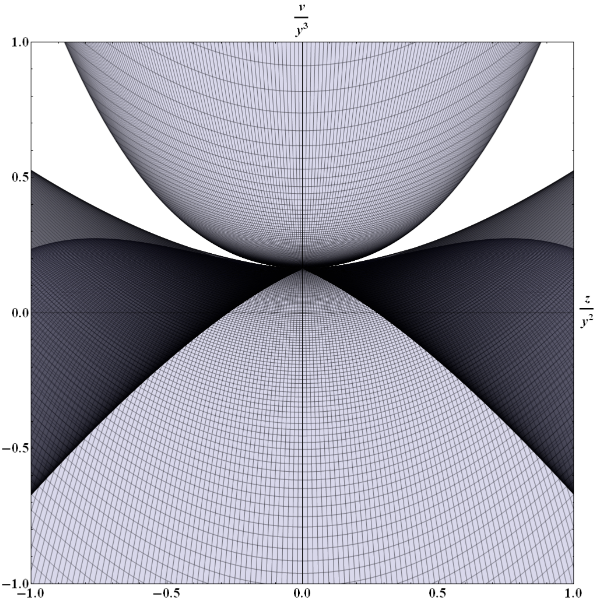
# Cut locus on the plane, where $x = 0$



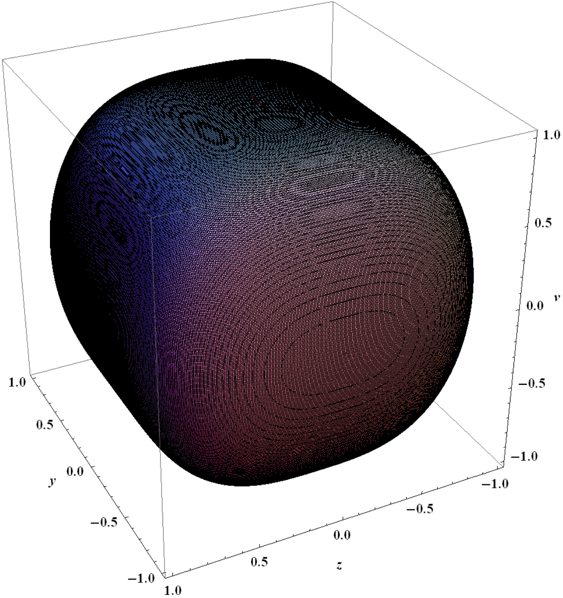
# Cut locus on the plane, where $x = 0$



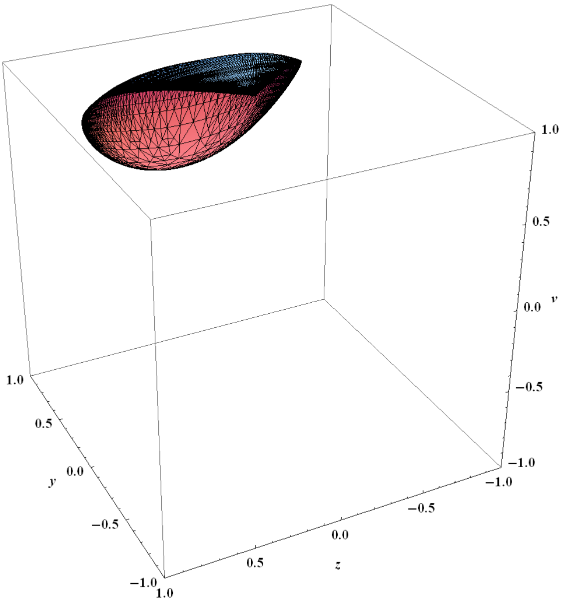
# Cut locus on the plane, where $x = 0$



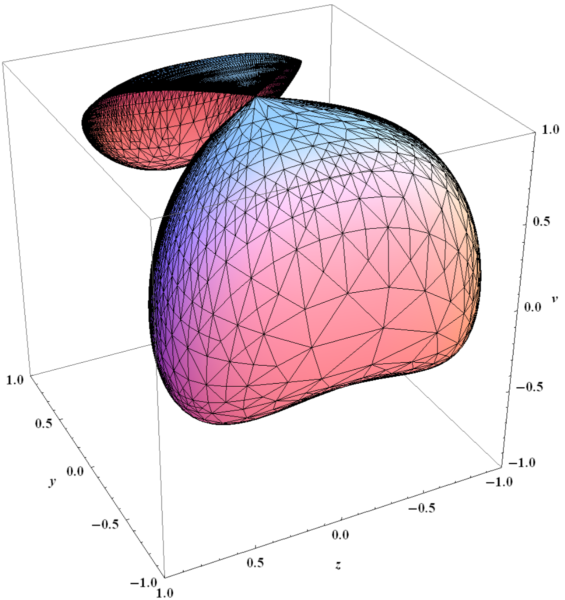
# Sphere $y^6 + |z|^3 + v^2 = 1$



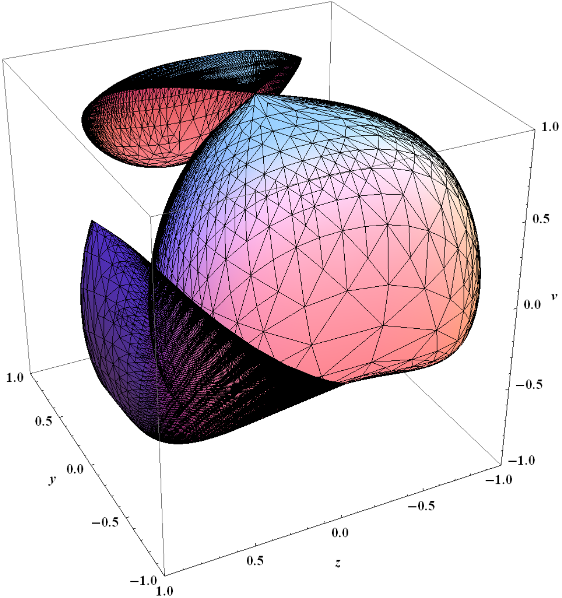
# Cut locus on the sphere, where $x = 0$



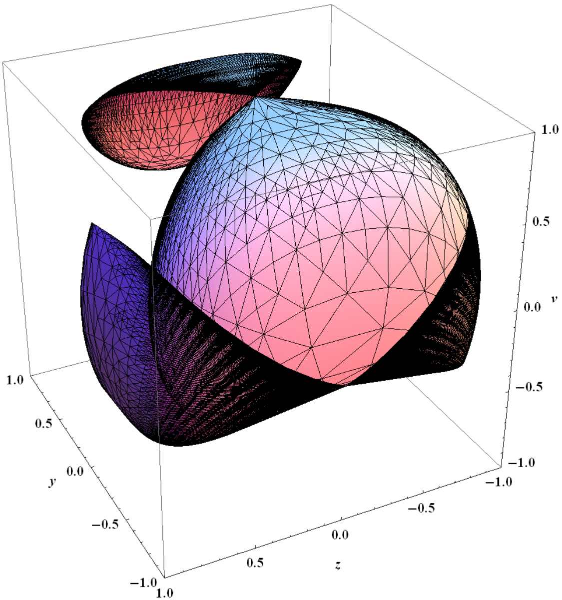
# Cut locus on the sphere, where $x = 0$



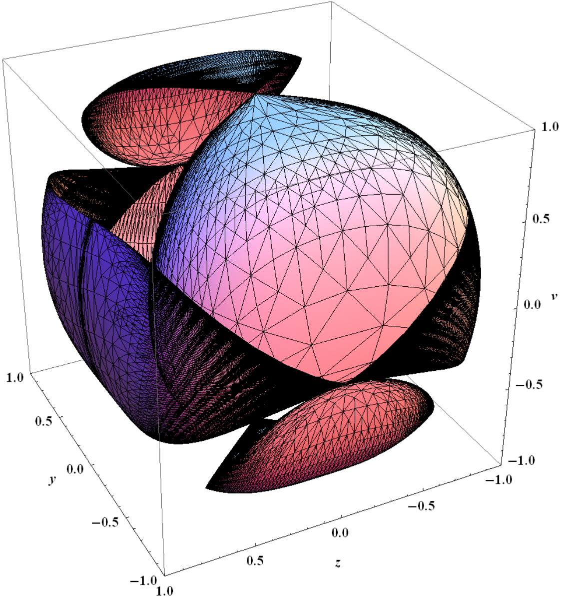
# Cut locus on the sphere, where $x = 0$



# Cut locus on the sphere, where $x = 0$



# Cut locus on the sphere, where $x = 0$



# Results

- Sub-Riemannian geodesics were found.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- The first Maxwell time was described. It was shown that this time coincide with the cut time.
- Finding of an optimal trajectory was was reduced to solving the system of algebraic equations in elliptic Jacobi functions and elliptic integrals.
- The program for computing optimal curves for the problem was developed for almost any boundary conditions.
- The description of cut locus was started.

# Plans

- Global structure of cut locus.
- Sub-Riemannian spheres and its singularities.
- Program for constructing all optimal solutions.
- Nilpotent approximation for a system modeling car with a trailer.

Thank you for your attention!