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CONTROL THEORY

Control Synthesis for a Three-Dimensional Nilpotent System

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Abstract— We consider the control problem for a system described by ordinary differential equations with linear controls. We present sufficient conditions for finding an exact solution of the control problem for a three-dimensional nilpotent system with a two-dimensional linear control in the form of programmed controls and feedback controls. We consider two examples of the computation of controls with the use of linear vector fields on the plane.

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1. INTRODUCTION

In the present paper, we find the solution of the control problem for a three-dimensional nilpotent system with a two-dimensional control and continue the previous research [1, 2]. In [1], we found formulas for programmed controls and feedback controls for a symmetric model of a nilpotent system in the class of controls optimal in the sense of the minimum of the functional of the sub-Riemannian length and in the classes of piecewise trigonometric and piecewise constant controls. In [2], we constructed a computational algorithm on the basis of the method of nilpotent approximation. Nilpotent approximations are analogs of linear approximations (see [3–7]) but, unlike the latter, preserve the controllability property, and the control problem for these systems has an exact solution in various classes of controls.

In the present paper, we develop a new approach to the solution of the control problem for a symmetric model of a nilpotent system, which substantially extends the possibilities of the control for three-dimensional nilpotent systems on the basis of geometric properties of a symmetric system. This approach provides the possibility to construct controls with given properties, which would be useful in the case of state space constraints. We present sufficient conditions that permit one to construct controls with the use of vector fields on the plane. Two algorithms are constructed for the solution of the control problem for a symmetric system, one of which is considered in detail in the case of a linear center-type field and is adapted to the case of a linear focus-type field. Linear fields are specified by four parameters; therefore, by varying these parameters, one can control the arrangement of trajectories joining edge points.

The controls obtained in the present paper completion the class of controls for the computational algorithm in [2], since the results obtained for the model system can be used for all nilpotent approximations. The controls obtained in the present paper permit one to extend the class of control problems; for example, state space constraints arise in the construction of maps in problems on three-dimensional manifolds.

2. STATEMENT OF THE PROBLEM

Consider the control system

$$\dot{z}_1 = u_1, \quad \dot{z}_2 = u_2, \quad \dot{z}_3 = (u_2 z_1 - u_1 z_2)/2, \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad z = (z_1, z_2, z_3) \in \mathbb{R}^3, \tag{1}$$

with the boundary conditions

$$z(0) = z^0, \qquad z(T) = 0, \qquad z^0 \in \mathbb{R}^3, \qquad T > 0. \tag{2}$$

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For this system, we pose the following control problem: for a given point $z^0 \in \mathbb{R}^3$, find a time $T > 0$ and piecewise continuous controls $u_i : [0,T] \to \mathbb{R}$, $i = 1, 2$, for which the corresponding trajectory $z(t)$ of system (1) satisfies condition (2).

System (1) is linear with respect to controls, has full rank, and hence is completely controllable (see [8, p. 73]); i.e., the control problem is solvable for each z^0.

The investigation of the control problem (1), (2) is the main aim of the present paper.

3. GEOMETRIC APPROACH TO THE SOLUTION OF THE CONTROL PROBLEM

In this section, we describe an approach to the solution of the control problem (1), (2) on the basis of geometric properties of trajectories of system (1).

3.1. Theoretical Information

Consider a simple closed piecewise smooth curve Γ on the plane Oz_1z_2. It is known that the area of the domain D bounded by the curve Γ is equal (neglecting the sign) to the integral

$$\frac{1}{2}\oint_\Gamma (z_1\,dz_2 - z_2\,dz_1). \tag{3}$$

The quantity (3) is referred to as the *algebraic area* of the domain D. If the contour Γ has the positive sense (i.e., is passed in the counterclockwise direction), then the algebraic area of the domain D is equal to the ordinary geometric area of D; and in the case of the negative sense of Γ, it is equal to the geometric area with the minus sign. If the closed curve Γ has self-intersections, then its algebraic area is evaluated as the sum of algebraic areas of subdomains bounded by simple closed curves in which Γ is split: the areas of subdomains passed around in the positive sense are taken with the plus sign, and those passed around in the negative direction, with the minus sign. If some of these subdomains are passed around several times, then the algebraic areas of those domains are accounted in the sum with coefficients equal to the number of passes.

Let γ be a smooth curve on the plane Oz_1z_2. The endpoints of the curve γ are joined with the origin O by rectilinear segments. The domain bounded by the resulting piecewise smooth contour Γ is referred to as a sector. Consider system (1) and an arbitrary initial state $z(0) = z^0$ of the system. Set $\bar{z}^0 = (z_1^0, z_2^0)$, Choose a smooth curve γ issuing from the point $P_0 = \bar{z}^0$, $\gamma = \{\bar{z}(t) |\ t \in [0, t_1]\}$, $\bar{z}(0) = \bar{z}^0$. Let $P = \bar{z}(t)$ be a current point of the curve γ. Then, by (3), the algebraic area of the sector OP_0P is equal to

$$S_{OP_0P} = \frac{1}{2}\int_\gamma (z_1\,dz_2 - z_2\,dz_1) = \frac{1}{2}\int_0^t (z_1\dot{z}_2 - z_2\dot{z}_1)\,dt \tag{4}$$

since the differential form $z_1\,dz_2 - z_2\,dz_1$ is zero on radius-segments.

3.2. Geometric Properties of Trajectories and Control Problems

Let $\bar{z}(t)$ be a parametric description of some smooth curve γ on the plane $\{z_3 = 0\}$, $\bar{z}(0) = \bar{z}^0$, where $\bar{z}^0$ is the projection of the initial state z^0 of system (1) onto the plane $\{z_3 = 0\}$. Set $u(t) = \dot{\bar{z}}(t)$; then $\bar{z}(t)$ is the solution of the Cauchy problem for the first two equations in system (1),

$$\dot{z}_1 = u_1(t), \qquad \dot{z}_2 = u_2(t), \tag{5}$$

with the initial condition

$$\bar{z}(0) = \bar{z}^0. \tag{6}$$

In this case, the Cauchy problem for the third equation in system (1) acquires the form

$$\dot{z}_3 = (z_1\dot{z}_2 - z_2\dot{z}_1)/2, \tag{7}$$

$$z_3(0) = z_3^0. \tag{8}$$

Let $z_3(t)$ be the solution of problem (7), (8). By comparing relations (4) and (7), we obtain

$$S_{OP_0P} = \int\limits_0^t \dot{z}_3(t)\,dt = z_3(t) - z_3(0) = \Delta z_3(t)|_\gamma. \tag{9}$$

Relation (9) clarifies the geometric property of system (1): if $z(t)$ is a solution of the Cauchy problem for system (1) with $z(0) = z^0$, then the increment $z_3(t) - z_3(0) = \Delta z_3(t)$ of the third coordinate along the projection $\bar{z}(t)$ of the curve $z(t)$ onto the plane Oz_1z_2 is equal to the algebraic area of the sector OP_0P. From relation (9), we obtain $z_3(t) = z_3^0 + S_{OP_0P}$, and the trajectory

$$z(t) = (\bar{z}(t), z_3^0 + S_{OP_0P}) \tag{10}$$

is a feasible trajectory of system (1).

In particular, is the curve γ is the radius segment $PO = \{\bar{\varrho}(t)|\ t \in [t_1, t_2]\}$, $P = \bar{\varrho}(t_1)$, then the curve $\bar{\varrho}(t)$ is the solution of the Cauchy problem (5), (6), and the third equation in system (1) acquires the form

$$\dot{z}_3(t) \equiv 0. \tag{11}$$

In this case, the trajectory

$$z(t) = (\bar{\varrho}(t), z_3^0) \tag{12}$$

is feasible for system (1).

Let us apply the geometric properties of the trajectories of system (1) to the solution of the control problem (1), (2). It follows from (10) and (11) that, to solve the posed problem, it suffices to choose a smooth curve γ on the plane Oz_1z_2 passing through the point P_0 and containing a point P_p such that the algebraic area of the sector OP_0P_p is equal to $-z_3^0$. Indeed, for the parameter value $t = t_p$ corresponding to the point P_p, the third coordinate of the trajectory (10) becomes zero. Then, by moving along the trajectory (12) starting from that time moment, the third coordinate of the trajectory remains identical zero by (11), and the first two coordinates achieve the origin at some time moment $T = t_p + T_1$. Such a displacement corresponds to controls of the form

$$u_i(t) = \begin{cases} \dot{z}_i(t) & \text{for} \quad t \in [0, t_p] \\ \dot{\varrho}_i(t) & \text{for} \quad t \in [t_p, T], \end{cases} \qquad i = 1, 2.$$

The following theorem provides sufficient conditions for the existence of such controls.

4. MAIN RESULT

Theorem 1. *Let $\bar{v}(z) = (v_1(z_1, z_2), v_2(z_1, z_2))$ be a smooth vector field on the plane Oz_1z_2 satisfying the following conditions.*

1. *$\bar{v}$ is a complete field; i.e., any trajectory $\bar{z}(t) = (z_1(t), z_2(t))$ of the autonomous differential equation*

$$\dot{z}_1 = v_1(z_1, z_2), \qquad \dot{z}_2 = v_2(z_1, z_2) \tag{13}$$

can be extended to the entire axis $t \in (-\infty, +\infty)$.

2. $\det(\bar{v}(\bar{z}), \bar{z}) \neq 0$ *for all $\bar{z} \neq 0$.*

3. *Both improper integrals $\int_0^{+\infty} (z_1\dot{z}_2 - z_2\dot{z}_1)\,dt$ and $\int_{-\infty}^0 (z_1\dot{z}_2 - z_2\dot{z}_1)\,dt$ are divergent.*

Then for any initial state $z^0 \in \mathbb{R}^3$, $(z_1^0)^2 + (z_2^0)^2 \neq 0$, there exists a unique time $t = t_p \geq 0$ such that the following programmed controls provide a solution of the control problem (1), (2)*:*

$$u_1(t) = \begin{cases} \pm v_1(\bar{z}(t)) & \textit{for} \quad t \in [0, t_p] \\ -\cos\varphi_p & \textit{for} \quad t \in [t_p, T], \end{cases} \qquad u_2(t) = \begin{cases} \pm v_2(\bar{z}(t)) & \textit{for} \quad t \in [0, t_p] \\ -\sin\varphi_p & \textit{for} \quad t \in [t_p, T], \end{cases} \tag{14}$$

where $\pm = \operatorname{sgn}(z_3^0\delta_{\bar{v}}^0)$; $\delta_{\bar{v}}^0 = \begin{vmatrix} v_1^0 & z_1^0 \\ v_2^0 & z_2^0 \end{vmatrix}$, $v_1^0 = v_1(\bar{z}^0)$, $v_2^0 = v_2(\bar{z}^0)$; *$\bar{z}(t)$ is the solution of the Cauchy problem* (13), (6); *(φ_p, r_p) are the polar coordinates of the point $\bar{z}^p = \bar{z}(\pm t_p)$; and $T = t_p + r_p$ is the total time of motion.*

Proof. First, let us show that, by using any vector field on the plane satisfying assumptions 1–3 of the theorem, one can construct controls bringing system (1) from the point

$$z^0 \in \mathbb{R}^3, \qquad (z_1^0)^2 + (z_2^0)^2 \neq 0, \qquad z_3^0 \neq 0,$$

into some point of the plane $\{z_3 = 0\}$. Take an arbitrary fixed field $\bar{v}$ with this property.

It follows from assumption 2 of the theorem that the origin is the unique singular point of the field $\bar{v}$ on the plane. Indeed, if $\bar{z} \neq 0$, then $\bar{v}(\bar{z}) \neq 0$. Moreover, condition 2 implies that the vector field $\bar{v}(\bar{z})$ has a nonzero index at the origin; therefore, $\bar{v}(0) = 0$. Hence it follows that any trajectory of the field $\bar{v}$ issuing outside zero does not pass through the origin; therefore, the continuous function $\delta_{\bar{v},\bar{z}}(t) = \det(\bar{v}(\bar{z}(t)),\ \bar{z}(t)) = -(z_1\dot{z}_2 - z_2\dot{z}_1)(t)$ is nonzero for any $t \in \mathbb{R}$, where $\bar{z}(t)$ is the solution of the Cauchy problem (13), (6). Consequently, $\delta_{\bar{v},\bar{z}}(t)$ preserves its sign on the entire numerical axis, and $\operatorname{sgn}\delta_{\bar{v},\bar{z}}(t) = \operatorname{sgn}\delta_{\bar{v},\bar{z}}(0) = \operatorname{sgn}\delta_{\bar{v}}^0$.

Consider the function $\Phi(t) = -(1/2)\int_0^t \delta_{\bar{v},\bar{z}}(t)\,dt$. It follows from the condition $\operatorname{sgn}\dot{\Phi}(t) = -\operatorname{sgn}\delta_{\bar{v}}^0$ for $t > 0$, $\Phi(0) = 0$, that the continuous function $\Phi(t)$ is strictly monotone and has constant sign for $t > 0$. This, together with assumption 3 of the theorem, implies the asymptotics

$$\lim_{t\to+\infty} \Phi(t) = \pm\infty, \qquad \pm = -\operatorname{sgn}\delta_{\bar{v}}^0. \tag{15}$$

Consider the field $-\bar{v}(\bar{z})$; then the solution of the Cauchy problem $\dot{\bar{z}} = -\bar{v}(\bar{z})$ with the initial condition (6) is given by the function

$$\bar{z}^-(t) = \bar{z}(-t) \quad \text{for any} \quad t \in \mathbb{R}, \tag{16}$$

where $\bar{z}(t)$ is the solution of the Cauchy problem (13), (6). By taking into account relation (16), one can show that

$$\delta_{-\bar{v},\bar{z}^-}(t) = \det(-\bar{v}(\bar{z}^-(t)), \bar{z}^-(t)) = -\delta_{\bar{v},\bar{z}}(-t), \tag{17}$$

which implies that $\operatorname{sgn}\delta_{-\bar{v},\bar{z}^-}(t) = \operatorname{sgn}\delta_{-\bar{v}}^0 = -\operatorname{sgn}\delta_{\bar{v}}^0$.

Consider the function $\Psi(t) = -(1/2)\int_0^t \delta_{-\bar{v},\bar{z}^-}(t)\,dt$. The condition $\operatorname{sgn}\dot{\Psi}(t) = \operatorname{sgn}\delta_{\bar{v}}^0$ with $t > 0$ and $\Psi(0) = 0$ implies that the continuous function $\Psi(t)$ is strictly monotone and has constant sign for $t > 0$; it follows from (17) that $\Psi(t) = \Phi(-t)$ for $t > 0$. This, together with assumption 3 of the theorem, implies the asymptotics

$$\lim_{t\to+\infty} \Psi(t) = \mp\infty, \qquad \pm = -\operatorname{sgn}\delta_{\bar{v}}^0. \tag{18}$$

Consider the equation $\Phi(t) = -z_3^0$. It follows from (15) that if $Sg = \operatorname{sgn}(\delta_{\bar{v}}^0 z_3^0) = 1$, then this equation has a unique solution $t = t_p^+ > 0$. If $Sg = -1$, then from the relations $\operatorname{sgn}(\Phi(t)\Psi(t)) = -1$ for $t > 0$ and from (18), we find that the equation $\Psi(t) = -z_3^0$ has the unique solution $t = t_p^- > 0$.

The existence of either t_p^+ or t_p^- implies that either $z_3^+(t)|_{t_p^+} = z_3^0 + \Phi(t)|_{t_p^+} = 0$ or $z_3^-(t)|_{t_p^-} = z_3^0 + \Psi(t)|_{t_p^-} = z_3^0 + \Phi(-t)|_{t_p^-} = 0$. Hence it follows that if we bring system (1) along the trajectory $z(t) = (\bar{z}(t), z_3^+(t))$, $t \in [0, t_p^+]$, then it is brought into the point $(z_1^+, z_2^+, 0)$; either if we bring the system along the trajectory $z(t) = (\bar{z}^-(t), z_3^-(t))$, $t \in [0, t_p^-]$, then it is brought into the point $(z_1^-, z_2^-, 0)$. We denote the switching point by $t = t_p^\pm$. By using the vector field $\bar{v}$, we have thereby constructed the trajectory $z(t) = (\bar{z}(\pm t), z_3^\pm(t))$, $t > 0$, $\pm = Sg$ of system (1) issuing from a given initial point $z(0) = (\bar{z}^0, z_3^0) = z^0$, $(z_1^0)^2 + (z_2^0)^2 \neq 0$, $z_3^0 \neq 0$, and getting on the plane $\{z_3 = 0\}$ into the point $z^p = z(t_p^\pm) = (\bar{z}(\pm t_p^\pm), 0)$, $\pm = Sg$ [see (16)].

Set $t_p^\pm = t_p$. Since the radial segments of the plane $\{z_3 = 0\}$ are admissible trajectories of system (1), it follows that the resulting trajectory can be supplemented by the radius-segment issuing from the point z^p at time $t = t_p$. Thus, we obtain an admissible trajectory of the original system corresponding to the posed control problem (1), (2). If $z^0 \in \mathbb{R}^3$, $(z_1^0)^2 + (z_2^0)^2 \neq 0$, $z_3^0 = 0$, then $t_p = 0$, and the trajectory (12) on the plane $\{z_3 = 0\}$ is the desired trajectory. By a straightforward verification, one can show that the program controls (14) bring the original system along one of the two constructed trajectories from the original state lying outside the axis Oz_3 into the origin. The proof of the theorem is complete.

Remark 1. If the total motion time T in the control problem (1), (2) is fixed, then, by performing reparametrization of the current time t in the trajectory $\bar{z}(t)$, one can obtain controls providing a solution of the control problem with fixed time $T > 0$ (see [1]).

Corollary 1. *The programmed controls* (14) *generate the control synthesis in the domain* $\{(z_1, z_2, z_3) \in \mathbb{R}^3 |\ (z_1)^2 + (z_2)^2 \neq 0\}$:

$$u_1(z) = \begin{cases} \pm v_1(z) & \text{if} \quad z_3 \neq 0,\ \bar{z} \neq 0 \\ -\cos\varphi & \text{if} \quad z_3 = 0, \end{cases} \tag{19}$$

$$u_2(z) = \begin{cases} \pm v_2(z) & \text{if} \quad z_3 \neq 0,\ \bar{z} \neq 0 \\ -\sin\varphi & \text{if} \quad z_3 = 0, \end{cases} \tag{20}$$

where $\pm = \operatorname{sgn}(z_3 \det(\bar{v}(\bar{z}), \bar{z}))$.

Proof. Formulas (19) and (20) are obtained from the programmed controls (14) by the substitution of $t = 0$.

5. ALGORITHMS

The above-proved theorem is constructive. It can be used for stating Algorithm 1 of the construction of controls of the form (14). At the preliminary stage, we choose a smooth field $\bar{v}(\bar{z})$ on the plane, solve the Cauchy problem (13), (6), and verify assumptions 1–3 of the theorem.

Algorithm 1.

The **input data** are $z^0 \in \mathbb{R}^3 :\ (z_1^0)^2 + (z_2^0)^2 \neq 0$; $\bar{v}(\bar{z})$ satisfy the assumptions of the theorem.

1. If $z_3^0 = 0$, then the solution of the control problem (), (2) is the programmed control $u(t) = (-\cos\varphi_0, -\sin\varphi_0)$, $t \in [0, T]$, where (r_0, φ_0) are the polar coordinates of the point $\bar{z}^0 = (z_1^0, z_2^0)$, $T = r_0$, and the operation of the algorithm is terminated.

2. If $z_3^0 \neq 0$, then

2_1 the parameters $\delta_{\bar{v}}^0 = \det(\bar{v}(\bar{z}^0), \bar{z}^0)$ and $Sg = \operatorname{sgn}(\delta_{\bar{v}}^0 z_3^0)$ are computed;

2_2 the switching time t_p is found: $t_p = |\tau|$ is a solution of the equation

$$\int\limits_0^{\pm|\tau|} \delta_{\bar{v},\bar{z}}(t)\, dt = 2z_3^0, \qquad \pm = Sg; \tag{21}$$

2_3 the polar coordinates of the switching point are found:

$$\bar{z}^p = \bar{z}(\pm t_p) = (r_p, \varphi_p), \qquad \pm = Sg; \tag{22}$$

2_4 the total motion time is computed as $T = t_p + r_p$.

3. The solution of the control problem (1), (2) is given by the programmed control $u(t)$ of the form (14), and the operation of the algorithm is terminated.

The output data are $u(t)$, $t \in [0, T]$.

By using Algorithm 1, we construct Algorithm 2 of the computation of controls bringing system (1) from points of the axis Oz_3 into the origin.

Algorithm 2.

The input data $z^0 :\ (z_1^0)^2 + (z_2^0)^2 = 0$, $z_3^0 \neq 0$; $\bar{v}(\bar{z})$ satisfies the assumptions of the theorem.

1. Moving system (1) away from the axis Oz_3. In the plane $\{z_3 = z_3^0\}$, we choose an arbitrary direction $\bar{l} = (\cos\varphi, \sin\varphi)$, $\varphi \in [0, 2\pi] :$ with the use of the controls $u^1(t) = \bar{l}$, the system is brought along a feasible trajectory, that is, the ray $\{t\cos\varphi, t\sin\varphi, z_3^0\}$, $t = [0, T_1]$, from the original state

$z^0 = (0, 0, z_3^0)$ into the intermediate state $z^1 = (T_1 \cos\varphi, T_1 \sin\varphi, z_3^0)$, where T_1 is an arbitrary positive number.

2. Motion into the point O. We use Algorithm 1 to which the point z^1 is input as the initial state. The linear change of variables $t(s) = s - T_1$, $s \in [T_1, T + T_1]$, is performed in the resulting program controls $u^2(t)$.

3. The solution of the control problem (1), (2) is the program control

$$u_i(t) = \begin{cases} u_i^1(t) & \text{for} \quad t \in [0, T_1] \\ u_i^2(t) & \text{for} \quad t \in [T_1, T + T_1],\ i = 1, 2. \end{cases} \tag{23}$$

The output data are $u(t)$, $t \in [0, T + T_1]$.

The constructed algorithms permit one to bring system (1) from an arbitrary point of the state space into the origin. In addition, the algorithms permit one to control the configuration of system trajectories issuing from a given initial point and entering the origin, since an arbitrary field $\bar{v}$ satisfying the assumptions of the theorem can be fed to the algorithm.

Consider the simplest class of complete smooth vector fields on the plane, that is, linear fields. We obtain families of solutions of the control problem (1), (2) with the use of Algorithm 1.

6. CONSTRUCTION OF THE CONTROL WITH THE USE OF A LINEAR CENTER-TYPE FIELD

Consider the linear field $\bar{v} = (v_1(\bar{z}), v_2(\bar{z}))$ of the form $v_1 = az_1 + bz_2$, $v_2 = cz_1 - az_2$ satisfying the conditions

$$a, b, c \in \mathbb{R}, \qquad a^2 + b^2 + c^2 \neq 0, \qquad bc < 0, \qquad \Delta = \begin{vmatrix} a & b \\ c & -a \end{vmatrix} > 0. \tag{24}$$

The field $\bar{v}$ of the type (24) is complete and has the unique singular point $(0, 0)$ of the type of center,

$$\det(\bar{v}(\bar{z}), \bar{z}) \neq 0 \qquad \forall \bar{z} \neq 0,$$

since the quadratic form $bz_2^2 - cz_1^2 + 2az_1z_2$ is definite (by virtue of the condition $\Delta > 0$).

The solution of the Cauchy problem (13), (6) is given by the ellipsoid

$$z_i^0(t) = z_i^0 \cos(t\sqrt{\Delta}\,) + \frac{v_i^0}{\sqrt{\Delta}} \sin(t\sqrt{\Delta}\,), \qquad i = 1, 2, \tag{25}$$

where $v_1^0 = az_1^0 + bz_2^0$ and $v_2^0 = cz_1^0 - az_2^0$.

By straightforward computations, one can show that

$$z_1\dot{z}_2 - z_2\dot{z}_1 = -\delta_{\bar{v}}^0 = b(z_2^0)^2 - c(z_1^0)^2 + 2az_1^0z_2^0 \equiv \text{const}, \tag{26}$$

which readily implies the divergence of the integrals in assumption 3 of the theorem.

Thus, the linear center-type linear field (24) satisfies all assumptions of the theorem.

Let us apply Algorithm 1 to the solution of the control problem (1), (2).

The input data $z^0 \in \mathbb{R}^3$: $(z_1^0)^2 + (z_2^0)^2 \neq 0$, $z_3^0 \neq 0$; $\bar{v}(\bar{z})$ is a field of the type (24).

By taking into account (26), we compute the direction of the pass around the ellipsoid (25) $Sg = \pm = \operatorname{sgn}(bz_3^0)$. By solving Eq. (21) with the use of relation (26), we obtain $t_p = 2|z_3^0|/|\delta_{\bar{v}}^0|$. By computing the polar coordinates of the point $\bar{z}^p = \bar{z}(\pm t_p)$, we find the desired solution, which is a program control of the form (14):

$$u_1(t) = \begin{cases} \pm(v_1^0 \cos(t\sqrt{\Delta}\,) - z_1^0\sqrt{\Delta}\,\sin(t\sqrt{\Delta}\,)) & \text{for} \quad t \in [0, t_p] \\ -\cos\varphi_p & \text{for} \quad t \in [t_p, T], \end{cases} \tag{27}$$

$$u_2(t) = \begin{cases} \pm(v_2^0 \cos(t\sqrt{\Delta}\,) - z_2^0\sqrt{\Delta}\,\sin(t\sqrt{\Delta}\,)) & \text{for} \quad t \in [0, t_p] \\ -\sin\varphi_p & \text{for} \quad t \in [t_p, T], \end{cases} \tag{28}$$

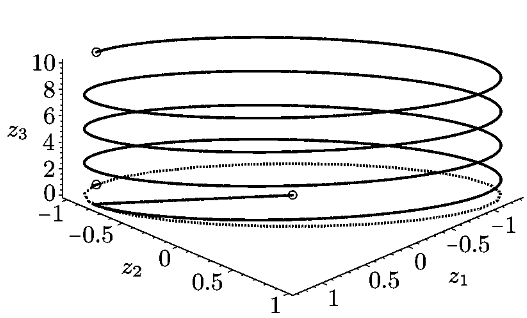

Fig. 1. Trajectory of the center type.

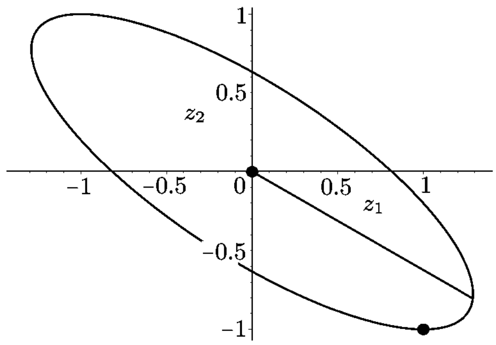

Fig. 2. Projection of a trajectory of the center type.

The control synthesis in the domain $\{(z_1, z_2, z_3) \in \mathbb{R}^3 |\ (z_1)^2 + (z_2)^2 \neq 0\}$ has the form

$$u_1(z) = \begin{cases} \pm v_1(z) & \text{if} \quad z_3 \neq 0,\ \bar{z} \neq 0 \\ -\cos\varphi & \text{if} \quad z_3 = 0, \end{cases} \qquad u_2(z) = \begin{cases} \pm v_2(z) & \text{if} \quad z_3 \neq 0,\ \bar{z} \neq 0 \\ -\sin\varphi & \text{if} \quad z_3 = 0, \end{cases} \tag{29}$$

where $\pm = \operatorname{sgn}(bz_3)$.

The trajectory that is a solution of the Cauchy problem for system (1) with the initial condition from (2) is the elliptic spiral $z(t) = (\bar{z}(t), z_3(t))$, where $\bar{z}(t)$ is the ellipsoid (25), and $z_3(t) = z_3^0 - (1/2)(\operatorname{sgn} z_3^0)|\delta_{\bar{v}}^0|t$ [see (26)].

As a special example, we choose a field $\bar{v}$ of the type (24): $a = 3$, $b = 5$, $c = -3$; the initial position is $z^0 = (1, -1, 10)$. Then the result of Algorithm 1 provides the trajectory shown in Fig. 1, and its projection onto the plane $\{z_3 = 0\}$ is presented in Fig. 2.

7. CONSTRUCTION OF CONTROL WITH THE USE OF A LINEAR FOCUS-TYPE FIELD

Consider the linear field $\bar{v} = (v_1(\bar{z}), v_2(\bar{z}))$ of the form $v_1 = az_1 + bz_2$, $v_2 = cz_1 + dz_2$ satisfying the conditions

$$a, b, c, d \in \mathbb{R}, \quad a^2 + b^2 + c^2 + d^2 \neq 0, \quad bc < 0, \quad S = a + d > 0, \quad \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} > S^2/4. \tag{30}$$

The field $\bar{v}$ of the type (30) is complete and has the unique singular point $(0, 0)$ of the focus type (the unstable focus if $S > 0$); $\det(\bar{v}(\bar{z}), \bar{z}) \neq 0$ for all $\bar{z} \neq 0$, since the quadratic form $bz_2^2 - cz_1^2 + (a - d)z_1 z_2$ is definite [$D = S^2 - 4\Delta < 0$ by virtue of condition (30)].

The solution of the Cauchy problem (13), (6) is given by the unwinding spiral

$$\begin{aligned} z_1(t) &= e^{St/2} \left(z_1^0 \cos(t\sqrt{|D|}/2) + \frac{(a-d)z_1^0 + 2bz_2^0}{\sqrt{|D|}} \sin(t\sqrt{|D|}/2) \right), \\ z_2(t) &= e^{St/2} \left(z_2^0 \cos(t\sqrt{|D|}/2) - \frac{(a-d)z_2^0 - 2cz_1^0}{\sqrt{|D|}} \sin(t\sqrt{|D|}/2) \right), \end{aligned} \tag{31}$$

where $D = S^2 - 4\Delta$ [see (30)].

By straightforward verifications, one can show that

$$z_1 \dot{z}_2 - z_2 \dot{z}_1 = -\delta_{\bar{v},\bar{z}}(t) = -e^{St}\delta_{\bar{v}}^0, \tag{32}$$

where $\delta^0_{\bar v} = b(z^0_2)^2 - c(z^0_1)^2 + (a-d)z^0_1 z^0_2$. Then, by taking into account the inequality $S > 0$, we obtain

$$\int_0^{+\infty} (z_1\dot z_2 - \dot z_1 z_2)\,dt = -\delta^0_{\bar v}\int_0^{+\infty} e^{St}\,dt = \pm\infty, \qquad \pm = -\operatorname{sgn}\delta^0_{\bar v}, \tag{33}$$

$$\int_{-\infty}^{0} (z_1\dot z_2 - \dot z_1 z_2)\,dt = -\delta^0_{\bar v}\int_{-\infty}^{0} e^{St}\,dt = \frac{-\delta^0_{\bar v}}{S} < \infty. \tag{34}$$

Thus, a linear field of the type (30) does not satisfy all assumptions of the theorem. It follows from relation (34) that, to construct sectors of an arbitrary given algebraic area, it is impossible to use the same trajectory of the field $\bar v$ but passed in the opposite direction; it can be used only in special cases in which $|z^0_3| < |\delta^0_{\bar v}|/(2S)$. Therefore, a focus-type focus can be used only in half-spaces. To perform the control in the whole state of states, one needs two focus-type field with opposite orientations of trajectories.

Consider two fields of the type (30): $\bar v = (v_1, v_2)$, $v_1 = az_1 + bz_2$, $v_2 = cz_1 + dz_2$ and $\bar w = (w_1, w_2)$, $w_1 = az_1 - bz_2$, $w_2 = -cz_1 + dz_2$. It is important that $\operatorname{sgn}\delta^0_{\bar v} = \operatorname{sgn} b = -\operatorname{sgn}\delta^0_{\bar w}$. Therefore, by the theorem, the sign of the expression bz^0_3 uniquely specifies the field that can be used for the control: the field $\bar v$ can be used in the half-space $\operatorname{sgn} z^0_3 = \operatorname{sgn} b$, and the field $\bar w$, in the half-space $\operatorname{sgn} z^0_3 = -\operatorname{sgn} b$.

Let us apply Algorithm 1 to the solution of the control problem (1), (2).

The input data are $z^0 \in \mathbb{R}^3$, $(z^0_1)^2 + (z^0_2)^2 \neq 0$, $z^0_3 \neq 0$; $\bar v(\bar z)$ is a field of the type (30).

1. Computation of the characteristics $Sg = \pm = \operatorname{sgn}(bz^0_3)$ of the problem.
2. Choice of either the field $\bar v$ or the field $\bar w$ $(az_1 \pm bz_2, \mp cz_1 + dz_2)$.
3. Choice of a trajectory of the Cauchy problem (13), (6) for the corresponding field $\bar v$ or $\bar w$:

$$\begin{aligned} z_1(t) &= e^{St/2}\left(z^0_1\cos(t\sqrt{|D|}/2) + \frac{(a-d)z^0_1 \pm 2bz^0_2}{\sqrt{|D|}}\sin(t\sqrt{|D|}/2)\right),\\ z_2(t) &= e^{St/2}\left(z^0_2\cos(t\sqrt{|D|}/2) - \frac{(a-d)z^0_2 \mp 2cz^0_1}{\sqrt{|D|}}\sin(t\sqrt{|D|}/2)\right), \end{aligned} \tag{35}$$

where $D = S^2 - 4\Delta$ [see (30)].

4. Computation of the characteristic $\delta^0 = \pm(b(z^0_2)^2 - c(z^0_1)^2) + (a-d)z^0_1 z^0_2$ of the trajectory.
5. Computation of the switching time: by solving Eq. (21) under condition (32) and by taking into account the relation $\operatorname{sgn}\delta^0 = \operatorname{sgn} z^0_3$, we obtain $t_p = (1/S)\ln(1 + 2Sz^0_3/\delta^0) > 0$.
6. Computation of the polar coordinates of the switching point and the total motion time: $\bar z^p = \bar z(t_p) = (r_p, \varphi_p)$ and $T = t_p + r_p$.
7. construction of program controls that give the solution of the control problem (1), (2):

$$u_1(t) = \begin{cases} az_1(t) \pm bz_2(t) & \text{for} \quad t \in [0, t_p] \\ -\cos\varphi_p & \text{for} \quad t \in [t_p, T], \end{cases} \tag{36}$$

$$u_2(t) = \begin{cases} \mp cz_1(t) + dz_2(t) & \text{for} \quad t \in [0, t_p] \\ -\sin\varphi_p & \text{for} \quad t \in [t_p, T]. \end{cases} \tag{37}$$

The output data is the control $u(t)$, $t \in [0, T]$.

Remark 2. The control synthesis in the domain $\{(z_1, z_2, z_3) \in \mathbb{R}^3 |\ (z_1)^2 + (z_2)^2 \neq 0\}$ has the form

$$u_1(z) = \begin{cases} az_1 + \beta z_2 & \text{if} \quad z_3 \neq 0,\ \bar z \neq 0 \\ -\cos\varphi & \text{if} \quad z_3 = 0, \end{cases} \qquad u_2(z) = \begin{cases} \gamma z_1 + dz_1 & \text{if} \quad z_3 \neq 0,\ \bar z \neq 0 \\ -\sin\varphi & \text{if} \quad z_3 = 0, \end{cases} \tag{38}$$

where $\beta = (\operatorname{sgn} z_3)|b|$, $\gamma = -(\operatorname{sgn} z_3)|c|$; a, β, γ, and d satisfy condition (30).

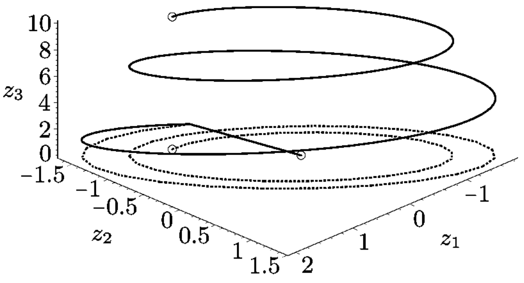

Fig. 3. A trajectory of the focus type.

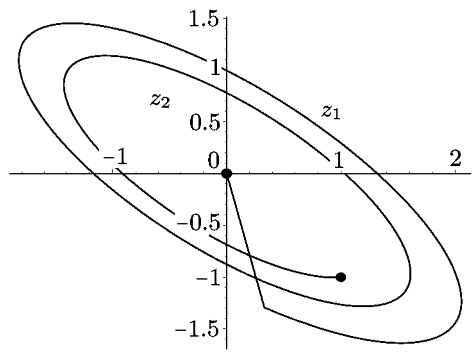

Fig. 4. The projection of a trajectory of the focus type.

The trajectory of the Cauchy problem for the original system has the form $z(t) = (\bar{z}(t), z_3(t))$, where $\bar{z}(t)$ is the trajectory (35) and $z_3^0(t) = -(\delta^0/2S)(e^{St} - 1) + z_3^0$.

As a special example, we take the field $\bar{v}$ of the type (30) with $a = 3$, $b = 5$, $c = -3$, and $d = 2.8$; the initial position is $z^0 = (1, -1, 10)$. Then the result of the operation of Algorithm 1 is the trajectory shown in Fig. 3, and its projection on the plane $\{z_3 = 0\}$ is represented to Fig. 4.

Note that the resulting algorithms for solving the control problem (1), (2) are implemented in the Maple software package (see [9]) and have been tested on particular linear fields of the types of center and focus. Various configurations of trajectories joining the origin with the initial point of system (1) were obtained.

The geometric approach to the solution of the control problems (1), (2) permits one to solve the problem for arbitrary boundary conditions with the choice of a curve γ of a given configuration with regard of state space constraints.

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