

MAXWELL STRATA AND CONJUGATE POINTS IN THE SUB-RIEMANNIAN PROBLEM ON THE LIE GROUP $SH(2)$

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Abstract. We study local and global optimality of geodesics in the left invariant sub-Riemannian problem on the Lie group $SH(2)$. We obtain the complete description of the Maxwell points corresponding to the discrete symmetries of the vertical subsystem of the Hamiltonian system. An effective upper bound on the cut time is obtained in terms of the first Maxwell times. We study the local optimality of extremal trajectories and prove the lower and upper bounds on the first conjugate times. We also obtain the generic time interval for the n -th conjugate time which is important in the study of sub-Riemannian wavefront.

Key words. Sub-Riemannian geometry, Special hyperbolic group $SH(2)$, Maxwell points, Cut time, Conjugate time

AMS subject classifications. 49J15, 93B27, 93C10, 53C17, 22E30

1. Introduction. Geometric control theory for linear systems was initiated in 1970s [1] and was extended to nonlinear systems in 1980s [2]. An important class of problems addressed by geometric control theory consists of control of the dynamical systems subjected to nonholonomic constraints [3],[4],[5]. It turns out that the optimal control of a large number of these physically interesting systems reduces to finding geodesics with respect to a sub-Riemannian metric [5]. Owing to the motivations and ramifications of sub-Riemannian problems in control theory, research on the sub-Riemannian problem on the group of motions of pseudo Euclidean plane was initiated in [6]. Motions of the pseudo Euclidean plane form the Lie group $SH(2)$ [7]. The sub-Riemannian problem on $SH(2)$ seeks to obtain optimal control for the system that comprises left invariant vector fields with 2-dimensional linear control input and energy cost functional. The study of sub-Riemannian problem on $SH(2)$ bears significance in the program of complete study of all the left-invariant sub-Riemannian problems on 3-dimensional Lie groups following the classification in terms of the basic differential invariants [8]. The Lie group $SH(2)$ gives one of the Thurston's 3-dimensional geometries called Sol [9].

In [6] parametrization of extremal trajectories in sub-Riemannian problem on the group $SH(2)$ was obtained via application of Pontryagin Maximum Principle (PMP). Since PMP provides only the necessary conditions for the optimal trajectories, the optimality conditions for given boundary points are therefore satisfied by a countable number of competing curves with different integral cost, not because of the optimality, but because the curves terminate on the boundary of the extended attainable set [10]. Second order and global optimality conditions such as conjugate points and Maxwell points are therefore investigated to establish optimality.

This paper is an extension of [6] in which we obtained complete parametrization of extremal trajectories $q_t = (x_t, y_t, z_t)$ and stated the general conditions for existence of the Maxwell points in terms of the equations $R_i(q_t) = 0$ and $z_t = 0$ (the functions R_i are given below in (2.16)). We now extend our analysis to completely characterize the Maxwell points and obtain the first Maxwell times. The first Maxwell time forms an effective upper bound on the cut time. We then investigate local optimality of the geodesics via description of conjugate points. The roots of the Jacobian of the exponential mapping are studied, and lower and upper bounds on the first conjugate time as well as the n -th conjugate time are obtained. We show that the function that gives the upper bound on the cut time provides the lower bound of the first conjugate time.

The paper is organized as follows. Section 2 presents a brief review of our results from [6]. In Section 3, we describe the roots of the functions $R_i(q_t)$ and z_t . These roots allow us to calculate the first Maxwell times and an effective upper bound on the cut time. Section 4 pertains to the local optimality analysis of geodesics via description of conjugate points. We compute the lower and upper bound on the first conjugate time as well as the bounds on the n -th conjugate time. Section 4 ends with the 3-dimensional plots of sub-Riemannian wavefront and sub-Riemannian spheres. Sections 5 and 6 pertain to future work and conclusion respectively.

2. Previous Work.

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2.1. Problem Statement. Motions of the pseudo Euclidean plane are distance and orientation preserving maps of the hyperbolic plane. These motions describe hyperbolic roto-translations of the pseudo Euclidean plane and form a 3-dimensional Lie group known as the special hyperbolic group $SH(2)$ [7]. Physically, the control system may be viewed as a unicycle moving on the hyperbolic plane. It is well known that a large class of systems in robotics such as differential drive robots, Reeds-Shepp car and even aircraft can be modeled as a unicycle [11] and therefore, we expect the results of this research to have potential impact in the field of robotics. The optimal control problem of these systems on the hyperbolic plane can be modeled as the sub-Riemannian problem on $SH(2)$ which reads as follows [6]:

$$\dot{x} = u_1 \cosh z, \quad \dot{y} = u_1 \sinh z, \quad \dot{z} = u_2, \quad (2.1)$$

$$q = (x, y, z) \in M = SH(2) \cong \mathbb{R}^3, \quad x, y, z \in \mathbb{R}, \quad (u_1, u_2) \in \mathbb{R}^2, \quad (2.2)$$

$$q(0) = (0, 0, 0), \quad q(t_1) = q_1 = (x_1, y_1, z_1), \quad (2.3)$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min. \quad (2.4)$$

By Cauchy-Schwarz inequality, the sub-Riemannian length functional l minimization problem (2.4) is equivalent to the problem of minimizing the following action functional with fixed t_1 [12]:

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2) dt \rightarrow \min. \quad (2.5)$$

2.2. Known Results. We now briefly review the results from [6] as a ready reference in this paper. System (2.1) satisfies the bracket generating condition and is hence globally controllable [13],[14]. Existence of optimal trajectories for the optimal control problem (2.1)–(2.5) follows from Filippov's theorem [4]. Since the problem is 3-dimensional contact, it is well known that abnormal extremal trajectories are constant [15]. We applied PMP [4] to (2.1)–(2.5) to derive the normal Hamiltonian system. It turns out that the vertical part of the normal Hamiltonian system is a double covering of a mathematical pendulum. The normal Hamiltonian system is given as:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin \gamma, \quad \lambda = (\gamma, c) \in C \cong (2S_\gamma^1) \times \mathbb{R}_c, \quad 2S_\gamma^1 = \mathbb{R}/(4\pi\mathbb{Z}), \quad (2.6)$$

$$\dot{x} = \cos \frac{\gamma}{2} \cosh z, \quad \dot{y} = \cos \frac{\gamma}{2} \sinh z, \quad \dot{z} = \sin \frac{\gamma}{2}. \quad (2.7)$$

The initial cylinder of the vertical subsystem was decomposed into the following subsets based upon the pendulum energy that correspond to various pendulum trajectories:

$$C = \bigcup_{i=1}^5 C_i,$$

where,

$$C_1 = \{\lambda \in C \mid E \in (-1, 1)\}, \quad (2.8)$$

$$C_2 = \{\lambda \in C \mid E \in (1, \infty)\}, \quad (2.9)$$

$$C_3 = \{\lambda \in C \mid E = 1, c \neq 0\}, \quad (2.10)$$

$$C_4 = \{\lambda \in C \mid E = -1, c = 0\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n, c = 0\}, \quad n \in \mathbb{N}, \quad (2.11)$$

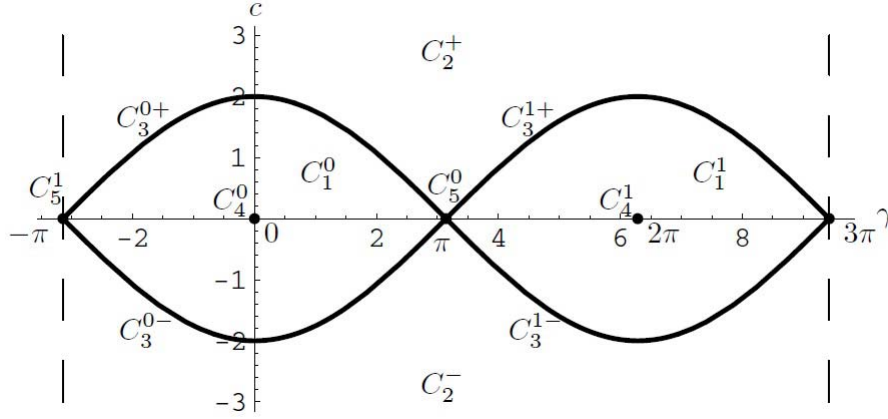
$$C_5 = \{\lambda \in C \mid E = 1, c = 0\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n + \pi, c = 0\}, \quad n \in \mathbb{N}. \quad (2.12)$$

We defined elliptic coordinates (φ, k) for $\lambda \in \cup_{i=1}^3 C_i \subset C$ and proved that the flow of the pendulum is rectified in these coordinates. Note that k was defined as the reparametrized energy and φ was defined as the reparametrized time of motion of the pendulum [6]. Integration of the horizontal subsystem in elliptic coordinates follows from integration of the vertical subsystem and the resulting extremal trajectories are parametrized by the Jacobi elliptic functions $\text{sn}(\varphi, k)$, $\text{cn}(\varphi, k)$, $\text{dn}(\varphi, k)$, $E(\varphi, k) = \int_0^\varphi \text{dn}^2(t, k) dt$ (Theorems 5.1-5.5 [6]). The results of integration for $\lambda \in C_i$, $i = 1, 2, 3$, are summarized as:

- Case 1 : $\lambda = (\varphi, k) \in C_1$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2} \left[\left(w + \frac{1}{w(1-k^2)} \right) [E(\varphi_t) - E(\varphi)] + \left(\frac{k}{w(1-k^2)} - kw \right) [\text{sn } \varphi_t - \text{sn } \varphi] \right] \\ \frac{1}{2} \left[\left(w - \frac{1}{w(1-k^2)} \right) [E(\varphi_t) - E(\varphi)] - \left(\frac{k}{w(1-k^2)} + kw \right) [\text{sn } \varphi_t - \text{sn } \varphi] \right] \\ s_1 \ln [(\text{dn } \varphi_t - k \text{cn } \varphi_t) \cdot w] \end{pmatrix}, \quad (2.13)$$

where $w = \frac{1}{\text{dn } \varphi - k \text{cn } \varphi}$, $s_1 = \text{sgn}(\cos \frac{\gamma}{2})$.

FIG. 2.1. Decomposition of the Phase Cylinder C of the Pendulum

- Case 2 : $\lambda = (\psi, k) \in C_2$

$$\begin{aligned}
 x_t &= \frac{1}{2} \left(\frac{1}{w(1-k^2)} - w \right) [E(\psi_t) - E(\psi) - k'^2 (\psi_t - \psi)] \\
 &\quad + \frac{1}{2} \left(kw + \frac{k}{w(1-k^2)} \right) [\text{sn } \psi_t - \text{sn } \psi], \\
 y_t &= -\frac{s_2}{2} \left(\frac{1}{w(1-k^2)} + w \right) [E(\psi_t) - E(\psi) - k'^2 (\psi_t - \psi)] \\
 &\quad + \frac{s_2}{2} \left(kw - \frac{k}{w(1-k^2)} \right) [\text{sn } \psi_t - \text{sn } \psi], \\
 z_t &= s_2 \ln[(\text{dn } \psi_t - k \text{cn } \psi_t) \cdot w],
 \end{aligned} \tag{2.14}$$

where $\psi = \frac{\varphi}{k}$, $\psi_t = \frac{\varphi_t}{k} = \psi + \frac{t}{k}$ and $w = \frac{1}{\text{dn } \psi - k \text{cn } \psi}$, $s_2 = \text{sgn } c$.

- Case 3 : $\lambda = (\varphi, k) \in C_3$

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2} \left[\frac{1}{w} (\varphi_t - \varphi) + w (\tanh \varphi_t - \tanh \varphi) \right] \\ \frac{s_2}{2} \left[\frac{1}{w} (\varphi_t - \varphi) - w (\tanh \varphi_t - \tanh \varphi) \right] \\ -s_1 s_2 \ln[w \text{sech } \varphi_t] \end{pmatrix}, \tag{2.15}$$

where $w = \cosh \varphi$.

The phase portrait of the pendulum admits a discrete group of symmetries $G = \{Id, \varepsilon^1, \dots, \varepsilon^7\}$. The symmetries ε^i are reflections and translations about the coordinates axes (γ, c) . These symmetries are exploited to state the general conditions on Maxwell strata in terms of the functions z_t and R_i given as:

$$R_1 = y \cosh \frac{z}{2} - x \sinh \frac{z}{2}, \quad R_2 = x \cosh \frac{z}{2} - y \sinh \frac{z}{2}. \tag{2.16}$$

We define the Maxwell sets MAX^i , $i = 1, \dots, 7$, resulting from the reflections ε^i of the extremals in the preimage of the exponential mapping N as:

$$MAX^i = \{ \nu = (\lambda, t) \in N \mid \lambda \neq \lambda^i, \quad \text{Exp}(\lambda, t) = \text{Exp}(\lambda^i, t) \},$$

where $\lambda = \varepsilon^i(\lambda)$. The corresponding Maxwell strata in the image of the exponential mapping are defined as:

$$Max^i = \text{Exp}(MAX^i) \subset M.$$

General description of the Maxwell strata is then given as:

$$\begin{aligned}
(1) \quad \nu \in MAX^1 &\Leftrightarrow \left\{ \begin{array}{ll} R_1(q) = 0, & \text{cn}\tau \neq 0, \quad \text{for } \lambda \in C_1 \\ R_1(q) = 0, & \text{for } \lambda \in C_2 \cup C_3 \end{array} \right\}, \\
(2) \quad \nu \in MAX^2 &\Leftrightarrow \left\{ \begin{array}{ll} z = 0, & \text{sn}\tau \neq 0, \quad \text{for } \lambda \in C_1 \cup C_2 \\ z = 0, & \tau \neq 0, \quad \text{for } \lambda \in C_3 \end{array} \right\}, \\
(3) \quad \nu \in MAX^6 &\Leftrightarrow \left\{ \begin{array}{ll} R_2(q) = 0, & \text{cn}\tau \neq 0, \quad \text{for } \lambda \in C_2 \\ R_2(q) = 0, & \text{for } \lambda \in C_1 \cup C_3 \end{array} \right\},
\end{aligned}$$

where,

$$\tau = \frac{1}{2}(\varphi_t + \varphi), \quad p = \frac{t}{2} \quad \text{when } \nu = (\lambda, t) \in N_1 \cup N_3, \quad (2.17)$$

$$\tau = \frac{1}{2k}(\varphi_t + \varphi), \quad p = \frac{t}{2k} \quad \text{when } \nu = (\lambda, t) \in N_2. \quad (2.18)$$

3. Complete Description of the Maxwell Strata.

3.1. Roots of Equations $R_i(q_t) = 0$ and $z_t = 0$. We now study roots of the equations $R_i(q_t) = 0$ and $z_t = 0$ to describe the Maxwell strata in the sub-Riemannian problem on SH(2). The idea is to obtain parametrization of the roots in terms of τ and p defined in (2.17)–(2.18). Using the addition formulas for Jacobi elliptic functions we get the following representation of the functions along extremal trajectories:

Case 1 - $\lambda \in C_1$:

$$\varphi_t = \tau + p, \quad \varphi = \tau - p, \quad (3.1)$$

$$\sinh z_t = s_1 \frac{2k \operatorname{sn} p \operatorname{sn} \tau}{\Delta}, \quad (3.2)$$

$$\sinh \frac{z_t}{2} = s_1 \frac{k \operatorname{sn} p \operatorname{sn} \tau}{\sqrt{\Delta}}, \quad (3.3)$$

$$\cosh \frac{z_t}{2} = \frac{1}{\sqrt{\Delta}}, \quad (3.4)$$

$$R_1(q_t) = \frac{2k}{1 - k^2} \operatorname{cn} \tau f_1(p), \quad (3.5)$$

$$R_2(q_t) = \frac{2s_1}{1 - k^2} \operatorname{dn} \tau f_2(p), \quad (3.6)$$

where $\Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau$, $f_1(p) = \operatorname{cn} p \operatorname{E}(p) - \operatorname{sn} p \operatorname{dn} p$ and $f_2(p) = \operatorname{dn} p \operatorname{E}(p) - k^2 \operatorname{sn} p \operatorname{cn} p$.

Case 2 - $\lambda \in C_2$:

$$\frac{\varphi_t}{k} = \tau + p, \quad \frac{\varphi}{k} = \tau - p, \quad (3.7)$$

$$\sinh z_t = s_2 \frac{2k \operatorname{sn} p \operatorname{sn} \tau}{\Delta}, \quad (3.8)$$

$$\sinh \frac{z_t}{2} = s_2 \frac{k \operatorname{sn} p \operatorname{sn} \tau}{\sqrt{\Delta}}, \quad (3.9)$$

$$\cosh \frac{z_t}{2} = \frac{1}{\sqrt{\Delta}}, \quad (3.10)$$

$$R_1(q_t) = \frac{2s_2}{1 - k^2} \operatorname{dn} \tau f_3(p), \quad (3.11)$$

$$R_2(q_t) = \frac{2k}{1 - k^2} \operatorname{cn} \tau f_4(p), \quad (3.12)$$

where $f_3(p) = -\operatorname{dn} p \operatorname{E}(p) + p \operatorname{dn} p (1 - k^2) + k^2 \operatorname{sn} p \operatorname{cn} p$ and $f_4(p) = -\operatorname{cn} p \operatorname{E}(p) + p \operatorname{cn} p (1 - k^2) + \operatorname{sn} p \operatorname{dn} p$.

Case 3 - $\lambda \in C_3$:

$$\varphi_t = \tau + p, \varphi = \tau - p, \quad (3.13)$$

$$\sinh z = 2s_1 s_2 \frac{\sinh(\tau) \sinh(p) \cosh(\tau) \cosh(p)}{\Delta}, \quad (3.14)$$

$$\sinh \frac{z_t}{2} = s_1 s_2 \frac{\sinh(\tau) \sinh(p)}{\sqrt{\Delta}}, \quad (3.15)$$

$$\cosh \frac{z_t}{2} = \frac{\cosh(\tau) \cosh(p)}{\sqrt{\Delta}}, \quad (3.16)$$

$$R_1(q_t) = s_2 \frac{2p - \sinh 2p}{2\sqrt{\Delta}}, \quad (3.17)$$

$$R_2(q_t) = s_1 \frac{2p + \sinh 2p}{2\sqrt{\Delta}}, \quad (3.18)$$

where $\Delta = \cosh^2 \tau + \sinh^2 p$.

PROPOSITION 3.1. *Let $t > 0$.*

- (1) If $\lambda \in C_1$ then $z_t = 0 \iff p = 2Kn, \quad \text{sn} \tau = 0$.
- (2) If $\lambda \in C_2$ then $z_t = 0 \iff p = 2Kn, \quad \text{sn} \tau = 0$.
- (3) If $\lambda \in C_3$ then $z_t = 0 \iff p = 0, \quad \tau = 0$.

Proof. Item (1) follows from (3.2), item (2) from (3.8) and item (3) from (3.14). \square

PROPOSITION 3.2. *The function $f_1(p)$ has an infinite number of roots for any $k \in [0, 1)$ given as:*

$$p = p_1^n(k), \quad n \in \mathbb{Z}, \quad (3.19)$$

$$p_1^0 = 0, \quad (3.20)$$

$$p_1^{-n}(k) = -p_1^n(k). \quad (3.21)$$

Moreover, the positive roots admit the bound:

$$p_1^n(k) \in (2nK, (2n+1)K), \quad n \in \mathbb{N}, \quad k \in (0, 1). \quad (3.22)$$

Proof. Equalities (3.20)–(3.21) follow directly from the fact that $f_1(p)$ is odd.

To prove (3.22) consider the function $g_1(p) = f_1(p)/\text{cnp}$, which has the same roots as $f_1(p)$ and also:

$$\lim_{p \rightarrow (2n-1)K+} g_1(p) \rightarrow +\infty,$$

$$\lim_{p \rightarrow (2n+1)K-} g_1(p) \rightarrow -\infty,$$

$$g_1'(p) = -\frac{(1-k^2)\text{sn}^2 p}{\text{cn}^2 p} \leq 0.$$

Hence $g_1(p)$ is decreasing on the interval $((2n-1)K, (2n+1)K)$ approaching $\pm\infty$ on the boundaries of the interval. It follows that $g_1(p)$ and therefore $f_1(p)$ admit a unique root $p = p_1^n(k)$ in each interval $((2n-1)K, (2n+1)K)$. Since $g_1(2nK) > 0$, for $n \in \mathbb{N}$, therefore $p_1^n(k) \in (2nK, (2n+1)K)$. Plots of the functions $f_1(p)$ and $g_1(p)$ for $k = 0.9$ are given in Figure 3.1. \square

LEMMA 3.3. *The function $f_2(p)$ is positive for any $p > 0$ and $k \in (0, 1)$.*

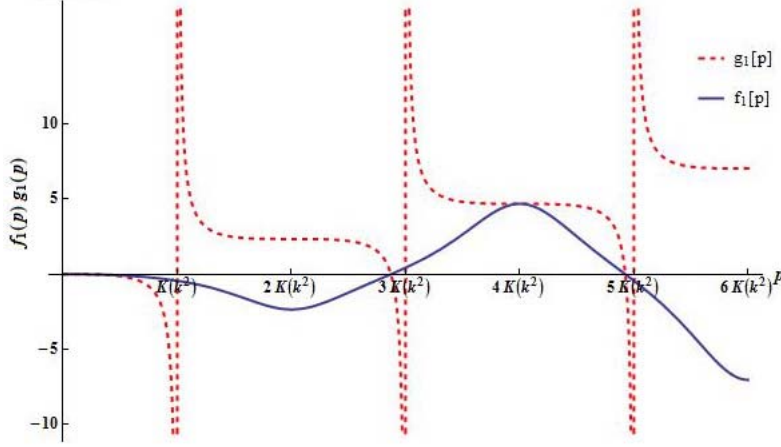
Proof. Consider the function $g_2(p) = f_2(p)/\text{dnp}$ where,

$$g_2'(p) = \frac{1-k^2}{\text{dn}^2 p} > 0.$$

Since $g_2(0) = 0$ therefore $g_2(p) > 0$ and $f_2(p) > 0$ for $p > 0$. \square

LEMMA 3.4. *The function $f_3(p)$ is negative for any $p > 0$ and $k \in (0, 1)$.*

Proof. Consider the function $g_3(p) = f_3(p)/\text{dnp}$ which has the same roots as $f_3(p)$ such that,

FIG. 3.1. Roots of the functions $f_1(p)$ and $g_1(p)$

$$g'_3(p) = -\frac{(1-k^2)k^2 \operatorname{sn}^2 p}{\operatorname{dn}^2 p} \leq 0.$$

Since $g_3(0) = 0$ therefore $g_3(p) < 0$ and $f_2(p) < 0$ for $p > 0$. \square

PROPOSITION 3.5. The function $f_4(p)$ has an infinite number of roots for any $k \in [0, 1)$ given as:

$$p = p_2^n(k), \quad n \in \mathbb{Z}, \quad (3.23)$$

$$p_2^0 = 0, \quad (3.24)$$

$$p_2^{-n}(k) = -p_2^n(k). \quad (3.25)$$

Moreover, the positive roots admit the bound:

$$p_2^n(k) \in (2nK, (2n+1)K), \quad n \in \mathbb{N}, \quad k \in (0, 1). \quad (3.26)$$

Proof. Equalities (3.24)–(3.25) follow directly from the fact that $f_4(p)$ is odd.

To prove (3.26) consider the function $g_4(p) = f_4(p)/\operatorname{cn} p$ which has the same roots as $f_4(p)$ and also:

$$\begin{aligned} \lim_{p \rightarrow (2n-1)K+} g_4(p) &\rightarrow -\infty, \\ \lim_{p \rightarrow (2n+1)K-} g_4(p) &\rightarrow +\infty, \\ g'_4(p) &= \frac{1-k^2}{\operatorname{cn}^2 p} > 0. \end{aligned}$$

Hence $g_4(p)$ is increasing on the interval $((2n-1)K, (2n+1)K)$ approaching $\mp\infty$ on the boundary of the interval. It follows that $g_4(p)$ and therefore $f_4(p)$ admits a unique root $p_2^n(k)$ on each such interval. Following an argument similar to the one in Proposition 3.2, it follows that $p_2^n(k) \in (2nK, (2n+1)K)$. Plots of the functions $f_4(p)$ and $g_4(p)$ for $k = 0.9$ are given in Figure 3.2. \square

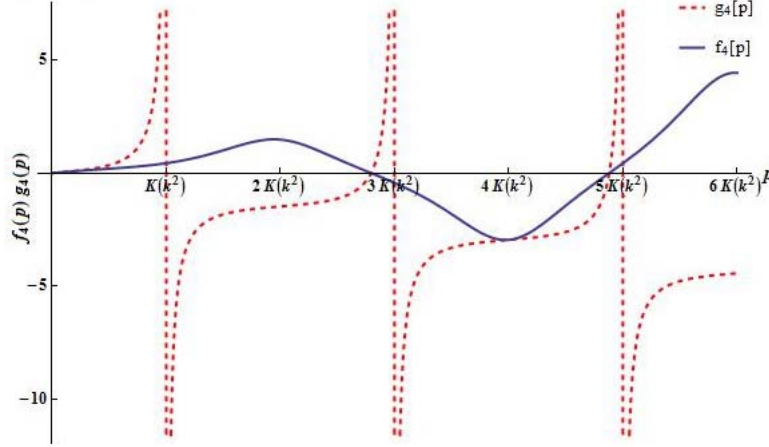
PROPOSITION 3.6. Let $t > 0$.

$$(1) \quad \text{If } \lambda \in C_1 \quad \text{then} \quad R_1(q_t) = 0 \iff p = p_1^n(k) \text{ or } \operatorname{cn} \tau = 0. \quad (3.27)$$

$$(2) \quad \text{If } \lambda \in C_2 \quad \text{then} \quad R_1(q_t) = 0 \text{ is impossible.} \quad (3.28)$$

$$(3) \quad \text{If } \lambda \in C_3 \quad \text{then} \quad R_1(q_t) = 0 \text{ is impossible.} \quad (3.29)$$

Proof. Item (1) follows from (3.5) and Proposition 3.2. Item (2) is given from (3.11) and Lemma 3.4. Item (3) follows from (3.17) where $2p - \sinh 2p = 0$ for $p = 0$ and $(2p - \sinh 2p)' = 2 - 2 \cosh 2p < 0$ for $p > 0$. Hence $R_1(q_t)$ does not admit any roots for $t > 0$ in this case. \square

FIG. 3.2. Roots of the functions $f_4(p)$ and $g_4(p)$

PROPOSITION 3.7. Let $t > 0$.

$$(1) \quad \text{If } \lambda \in C_1 \quad \text{then} \quad R_2(q_t) = 0 \quad \text{is impossible.} \quad (3.30)$$

$$(2) \quad \text{If } \lambda \in C_2 \quad \text{then} \quad R_2(q_t) = 0, \quad \Longleftrightarrow \quad p = p_2^n(k) \text{ or } \text{cn}\tau = 0. \quad (3.31)$$

$$(3) \quad \text{If } \lambda \in C_3 \quad \text{then} \quad R_2(q_t) = 0 \quad \text{is impossible.} \quad (3.32)$$

Proof. Item (1) is given from (3.6) and Lemma 3.3. Item (2) is given from (3.12) and Proposition 3.5. Item (3) follows from (3.18) where $2p + \sinh 2p = 0$ for $p = 0$ and $(2p + \sinh 2p)' = 2 + 2 \cosh 2p > 0$ for $p \geq 0$. Hence $R_2(q_t)$ does not admit any root for $t > 0$ in this case. \square Let us now summarize the results obtained on the characterization of Maxwell strata.

THEOREM 3.8. The Maxwell strata $MAX^i \cap N^j$ are given as:

- (1) $MAX^1 \cap N_1 = \{\nu \in N_1 \mid p = p_1^n(k), \quad n \in \mathbb{N}, \quad \text{cn}\tau \neq 0\},$
- (2) $MAX^1 \cap N_2 = MAX^1 \cap N_3 = \emptyset,$
- (3) $MAX^2 \cap N_1 = MAX^2 \cap N_2 = \{\nu \in N_1 \cup N_2 \mid p = 2nK(k), \quad n \in \mathbb{N}, \quad \text{sn}\tau \neq 0\},$
- (4) $MAX^2 \cap N_3 = \emptyset,$
- (5) $MAX^6 \cap N_1 = MAX^6 \cap N_3 = \emptyset,$
- (6) $MAX^6 \cap N_2 = \{\nu \in N_2 \mid p = p_2^n(k), \quad n \in \mathbb{N}, \quad \text{cn}\tau \neq 0\}.$

Proof. This follows from the general description of the Maxwell strata and the Propositions 3.1, 3.6 and 3.7. \square

3.2. Limit Points of the Maxwell Set. It remains to consider the points at the boundary of the Maxwell strata like the points in N_1 with $p = p_1^n(k)$, $\text{cn}\tau = 0$. Since the action of reflections in the preimage of exponential map is the same for SH(2) and SE(2), it can be readily seen using Proposition 5.8 [16] that when $\nu \in N_1$, $p = p_1^1(k)$, $\text{cn}\tau = 0$ and when $\nu \in N_2$, $p = p_2^1(k)$, $\text{cn}\tau = 0$ then $q_t = \text{Exp}(\nu)$ is a conjugate point. The same reasoning applies to the case when $\nu \in N_1$, $\text{sn}\tau = 0$ and $\nu \in N_2$, $\text{sn}\tau = 0$. Thus we get the following statement.

PROPOSITION 3.9. A point $q_t = \text{Exp}(\nu)$ is conjugate to the initial point q_0 if the following conditions hold:

- (1) $\nu \in N_1, \quad p = p_1^n(k), \quad n \in \mathbb{N}, \quad \text{cn}\tau = 0.$
- (2) $\nu \in N_1 \cup N_2, \quad p = 2nK(k), \quad n \in \mathbb{N}, \quad \text{sn}\tau = 0.$
- (3) $\nu \in N_2 = p = p_2^n(k), \quad n \in \mathbb{N}, \quad \text{cn}\tau = 0.$

3.3. Upper Bound on Cut Time. It is well known that a normal extremal trajectory cannot be optimal after the first Maxwell time. We now calculate the first Maxwell time $t_1^{MAX} : C \rightarrow (0, +\infty]$.

PROPOSITION 3.10. The first Maxwell time t_1^{MAX} corresponding to the reflections $\varepsilon^1, \varepsilon^2, \varepsilon^6$ is given as:

$$\begin{aligned} \lambda \in C_1 &\implies t_1^{MAX}(\lambda) = 4K(k), \\ \lambda \in C_2 &\implies t_1^{MAX}(\lambda) = 4kK(k), \\ \lambda \in C_3 \cup C_4 \cup C_5 &\implies t_1^{MAX}(\lambda) = +\infty. \end{aligned}$$

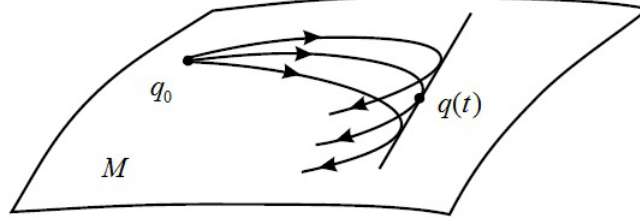


FIG. 4.1. Concept of conjugate point

Proof. For $\lambda \in C_1, C_2, C_3$ apply Theorem 3.8 and Proposition 3.9. For $\lambda \in C_4$ and $\lambda \in C_5$, apply Theorems 5.4, 5.5 and Proposition 6.3 from [6]. \square Using Proposition 3.10 we get the following global upper bound on the cut time $t_{\text{cut}}(\lambda)$ for extremal trajectories.

COROLLARY 3.11. For any $\lambda \in C$,

$$t_{\text{cut}}(\lambda) \leq t_1^{\text{MAX}}(\lambda). \quad (3.33)$$

We believe that inequality (3.33) is in fact an equality and plan to prove this conjecture in a forthcoming work.

4. Conjugate Points. In this section we study local optimality of sub-Riemannian geodesics and compute the first conjugate time (i.e., the time of loss of local optimality) along extremal trajectories. Let us recall certain important facts related to conjugate points which will also outline the scheme of further analysis. A point $q_t = \text{Exp}(\lambda, t)$ is called a conjugate point for q_0 if $\nu = (\lambda, t) = (\gamma, c, t)$ is a critical point of the exponential mapping, q_t being its critical value. In other words, this definition is given as:

$$d_\nu \text{Exp} : T_\nu N \rightarrow T_{q_t} M \text{ is degenerate,}$$

where $d_\nu \text{Exp}$ amounts to the Jacobian J of the exponential mapping i.e.,

$$J = \frac{\partial(x_t, y_t, z_t)}{\partial(\gamma, c, t)} = \begin{vmatrix} \frac{\partial x_t}{\partial \gamma} & \frac{\partial x_t}{\partial c} & \frac{\partial x_t}{\partial t} \\ \frac{\partial y_t}{\partial \gamma} & \frac{\partial y_t}{\partial c} & \frac{\partial y_t}{\partial t} \\ \frac{\partial z_t}{\partial \gamma} & \frac{\partial z_t}{\partial c} & \frac{\partial z_t}{\partial t} \end{vmatrix}.$$

According to the definition, roots of the equation $J = 0$ give the conjugate points and the time corresponding to these roots is called the conjugate time. Carl Gustav Jacob Jacobi (1804–1851) gave a geometric interpretation of conjugate points according to which a conjugate point q_t of a point q_0 is the point where an extremal trajectory meets the envelope of the set of extremal trajectories through q_0 [17]. This is depicted in Figure 4.1. In the local optimality analysis the first conjugate time is an important notion as this is the time at which an extremal trajectory loses local optimality. The first conjugate time is defined as:

$$t_1^{\text{conj}}(\lambda) = \inf \{ t > 0 \mid t \text{ is a conjugate time along } \text{Exp}(\lambda, s), \quad s \geq 0 \}.$$

4.1. Conjugate Points and Homotopy. A lower bound of the form $t_1^{\text{conj}}(\lambda) \geq t_1^{\text{MAX}}(\lambda)$ for all extremal trajectories $q(t) = \text{Exp}(\lambda, t)$ was proved in the Euler Elastic problem [18], sub-Riemannian problem on $\text{SE}(2)$ [19] and sub-Riemannian problem on the Engel group [20] via homotopy considering the fact that Maslov index (number of conjugate points along an extremal trajectory) is invariant under homotopy [21]. In order to qualify for proof of absence of conjugate points below the lower bound of the first conjugate time via homotopy, the optimal control problem must satisfy a set of hypotheses (H1)–(H4) [18] outlined below.

Consider a general analytic optimal control problem on an analytic manifold M :

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \quad (4.1)$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ is fixed}, \quad (4.2)$$

$$J = \int_0^{t_1} \Phi(q(t), u(t)) dt \rightarrow \min, \quad (4.3)$$

where $f(q, u)$ is a family of vector fields and $\Phi(q, u)$ is some function on M analytic in system state $q \in M$ and control parameter $u \in U$. Note that the sub-Riemannian problem on $M = \text{SH}(2)$ (2.1)–(2.5) is of this form. Let the control dependent normal Hamiltonian of PMP for (4.1)–(4.3) be given as:

$$h_u(\lambda) = \langle \lambda, f(q, u) \rangle - \Phi(q, u). \quad (4.4)$$

Let a triple $(\tilde{u}(t), \lambda_t, q(t))$ represent respectively the extremal control, extremal and extremal trajectory corresponding to the normal Hamiltonian $h_u(\lambda)$. Let the following hypotheses be satisfied for (4.1)–(4.3):

(H1) For all $\lambda \in T^*M$ and $u \in U$, the quadratic form $\frac{\partial^2 h_u}{\partial u^2}(\lambda)$ is negative definite. This is the strong Legendre condition along the extremal pair $(\tilde{u}(t), \lambda(t))$.

(H2) For any $\lambda \in T^*M$, the function $u \mapsto h_u(\lambda)$, $u \in U$, has a maximum point $\bar{u}(\lambda) \in U$:

$$h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \quad \lambda \in T^*M.$$

(H3) The extremal control $\tilde{u}(\cdot)$ is a corank one critical point of the endpoint mapping.

(H4) All trajectories of the Hamiltonian vector field $\vec{H}(\lambda)$, $\lambda \in T^*M$, are continued for $t \in [0, +\infty)$.

Under the hypotheses (H1)–(H4), the following is true for the optimal control problem of the form (4.1)–(4.3):

1. Normal extremal trajectories lose their local optimality (both strong and weak) at the first conjugate point, see [4].
2. Along each normal extremal trajectory, conjugate times are isolated one from another, see [18],[19].

We will apply the following statement for the proof of absence of conjugate points via homotopy.

PROPOSITION 4.1. (Corollaries 2.2 and 2.3 [18]). *Let $(u^s(t), \lambda_t^s)$, $t \in [0, +\infty)$, $s \in [0, 1]$, be continuous in parameter s family of normal extremal pairs in the optimal control problem (4.1)–(4.3) satisfying hypotheses (H1)–(H4). Let $s \mapsto t_1^s$ be a continuous function, $s \in [0, 1]$, $t_1^s \in (0, +\infty)$. Assume that for any $s \in [0, 1]$ the instant $t = t_1^s$ is not a conjugate time along the extremal λ_t^s . If the extremal trajectory $q^0(t) = \pi(\lambda_t^0)$, $t \in (0, t_1^0]$, does not contain conjugate points, then the extremal trajectory $q^1(t) = \pi(\lambda_t^1)$, $t \in (0, t_1^1]$, also does not contain conjugate points. It can be easily checked that the sub-Riemannian problem (2.1)–(2.5) satisfies the hypotheses (H1)–(H4) and therefore Proposition 4.1 can be used to prove bounds of the first conjugate time t_1^{conj} .*

4.2. Bounds on t_1^{conj} for $\lambda \in C_1$. Using the elliptic coordinates (φ, k) defined in Section 5.3.1 [6] and parametrization of extremal trajectories (2.8), the Jacobian of the exponential mapping is given as:

$$J = \frac{\partial(x_t, y_t, z_t)}{\partial(\varphi, k, t)} = \frac{J_1(p, \tau, k)}{(1 - k^2)^2(1 - k \text{sn} p \text{sn} \tau)^2}, \quad (4.5)$$

$$J_1(p, \tau, k) = -4k(\alpha_1 + \alpha_2 + \alpha_3), \quad (4.6)$$

$$\alpha_1(p, \tau, k) = \text{sn} p \text{cn} p \text{dn} p (2E(p) - p + k^2 p),$$

$$\alpha_2(p, \tau, k) = -\text{dn}^2 p \text{sn}^2 p - k^2 \text{sn}^2 p \text{cn}^2 \tau,$$

$$\alpha_3(p, \tau, k) = E(p) (\text{sn}^2 p - \text{sn}^2 \tau) (E(p) - p + k^2 p),$$

where p and τ for $\lambda \in C_1$ were defined in (2.17). Plots of $J_1(p, \tau, k)$ are shown in Figures 4.2, 4.3.

LEMMA 4.2. *There exists $\hat{k} \in (0, 1)$ such that for all $k \in (0, \hat{k})$ and $p \in (0, \pi)$, the function J_1 is positive.*

Proof. The Taylor expansions of J_1 are given as:

$$J_1 = 4k \sin p (-p \cos p + \sin p), \quad k \rightarrow 0, \quad (4.7)$$

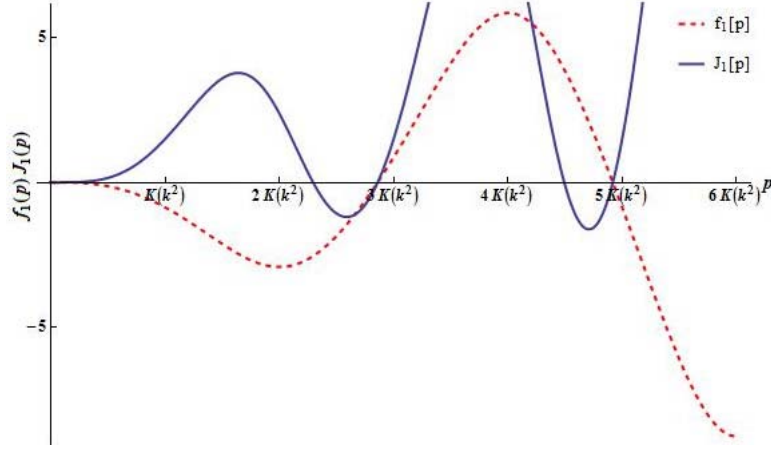
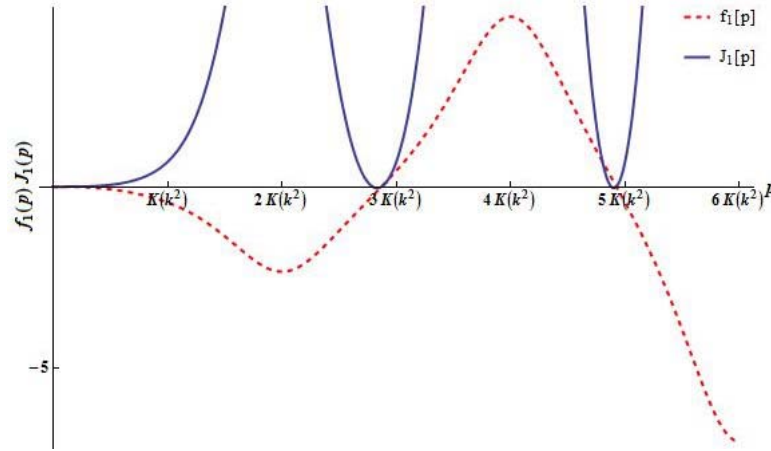
$$J_1 = \frac{4}{3} k p^4 + o(k^2 + p^2)^4, \quad k^2 + p^2 \rightarrow 0. \quad (4.8)$$

From (4.7) it can be readily seen that in limit passage of $k \rightarrow 0^+$, $J_1 > 0$ for $p \in (0, \pi)$. Note that $2K(0) = \pi$. Similarly from (4.8) it follows that $J_1 > 0$ when $k^2 + p^2 \rightarrow 0^+$. \square

LEMMA 4.3. *If $k \in (0, 1)$ and $p = 2nK(k)$ for $n \in \mathbb{Z}$, then $J_1 \geq 0$.*

Proof. Direct substitution of $p = 2nK(k)$ to (4.6) gives:

$$J_1 = 16n^2 k E(k) (E(k) - (1 - k^2)K(k)) \text{sn}^2 \tau. \quad (4.9)$$

FIG. 4.2. $J_1(p, \tau, k)$ and $f_1(p)$ for $k = 0.5$ FIG. 4.3. $J_1(p, \tau, k)$ and $f_1(p)$ for $k = 0.9$

Since $f(k) = E(k) - (1 - k^2)K(k) > 0$ because $f(0) = 0$ and $f'(k) = kK(k) > 0$, therefore, $J_1 \geq 0$. \square

LEMMA 4.4. *The system of equations*

$$f_1(p, k) = 0, \quad J = 0, \quad (4.10)$$

is incompatible for $k \in (0, 1)$, $p > 0$.

Proof. We denote

$$E(u, k) = \int_0^u \sqrt{1 - k^2 \sin^2 \varphi} d\varphi, \quad F(u, k) = \int_0^u \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

The system of equations (4.10), after the change $p = \text{am}(u, k)$, turns into:

$$\begin{cases} E(u, k) \cos u = \sqrt{1 - k^2 \sin^2 u} \sin u, \\ F(u, k) \sqrt{1 - k^2 \sin^2 u} \cos u = \sin u. \end{cases} \quad (4.11)$$

We prove that system (4.11) is incompatible for $k \in (0, 1)$, $u > 0$.

(1) Let $0 < u < \pi/2$. System (4.11) implies the equation:

$$\frac{E(u, k)}{\sqrt{1 - k^2 \sin^2 u}} = F(u, k) \sqrt{1 - k^2 \sin^2 u},$$

which is equivalent to the following equations:

$$\begin{aligned}
\int_0^u \frac{\sqrt{1-k^2 \sin^2 \varphi}}{\sqrt{1-k^2 \sin^2 u}} d\varphi &= \int_0^u \frac{\sqrt{1-k^2 \sin^2 u}}{\sqrt{1-k^2 \sin^2 \varphi}} d\varphi, \\
\int_0^u \left(\frac{\sqrt{1-k^2 \sin^2 \varphi}}{\sqrt{1-k^2 \sin^2 u}} - \frac{\sqrt{1-k^2 \sin^2 u}}{\sqrt{1-k^2 \sin^2 \varphi}} \right) d\varphi &= 0, \\
\int_0^u \frac{1-k^2 \sin^2 \varphi - (1-k^2 \sin^2 u)}{\sqrt{1-k^2 \sin^2 u} \sqrt{1-k^2 \sin^2 \varphi}} d\varphi &= 0, \\
\int_0^u \frac{\sin^2 u - \sin^2 \varphi}{1-k^2 \sin^2 \varphi} d\varphi &= 0.
\end{aligned}$$

The last equality is impossible since the function under the integral is positive for $0 < \pi < u$ (when $0 < u < \pi/2$).

(2) Equations of system (4.11) are violated when $\cos u = 0$ or $\sin u = 0$, i.e., at the points $u = \frac{\pi k}{2}$, $k \in \mathbb{N}$. This is checked immediately.

(3) For $\frac{\pi}{2} < u < \pi$ system (4.11) is incompatible since the function \cos is negative, while the functions \sin , E and F are positive.

(4) It remains to consider the case $u > \pi$ for $\sin u \cos u \neq 0$. In this case we multiply the equations of the system, divide the first equation by the second one, and get the following system:

$$\left\{ \begin{array}{l} \cos^2 u E(u, k) F(u, k) = \sin^2 u, \\ \frac{E(u, k)}{F(u, k) \sqrt{1-k^2 \sin^2 u}} = \sqrt{1-k^2 \sin^2 u}. \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} E(u, k) F(u, k) = \tan^2 u, \\ E(u, k) = F(u, k) (1-k^2 \sin^2 u). \end{array} \right.$$

The equality $1 + \tan^2 u = \cos^{-2} u$ and the equation $E(u, k) F(u, k) = \tan^2 u$ imply:

$$\cos^2 u = \frac{1}{1 + E(u, k) F(u, k)}.$$

Since $1 - k^2 \sin^2 u = 1 - k^2 + k^2 \cos^2 u = 1 - k^2 + \frac{k^2}{1 + E(u, k) F(u, k)}$, then the equation $E(u, k) = F(u, k) (1 - k^2 \sin^2 u)$ is rewritten as:

$$E(u, k) = F(u, k) (1 - k^2) + \frac{k^2 F(u, k)}{1 + E(u, k) F(u, k)}. \quad (4.12)$$

We have,

$$\begin{aligned}
E(u, k) - (1 - k^2) F(u, k) &= \int_0^u \left(\sqrt{1 - k^2 \sin^2 \varphi} - \frac{1 - k^2}{\sqrt{1 - k^2 \sin^2 \varphi}} \right) d\varphi, \\
&= \int_0^u \frac{1 - k^2 \sin^2 \varphi - (1 - k^2)}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi = \int_0^u \frac{k^2 - k^2 \sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi, \\
&= k^2 \int_0^u \frac{\cos^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi.
\end{aligned}$$

Consequently, equation (4.12) takes the form:

$$k^2 \int_0^u \frac{\cos^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi = \frac{k^2 F(u, k)}{1 + E(u, k) F(u, k)},$$

and after dividing both sides by k^2 we get:

$$\int_0^u \frac{\cos^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi = \frac{F(u, k)}{1 + E(u, k) F(u, k)}.$$

Since $\frac{1}{\sqrt{1-k^2 \sin^2 \varphi}} > 1$, $u > \pi$ then there hold the inequalities:

$$\frac{\cos^2 \varphi}{\sqrt{1-k^2 \sin^2 \varphi}} > \int_0^u \cos^2 \varphi d\varphi > \int_0^\pi \cos^2 \varphi d\varphi = \frac{\pi}{2}.$$

Consequently,

$$\frac{F(u, k)}{1 + E(u, k)F(u, k)} = \int_0^u \frac{\cos^2 \varphi}{\sqrt{1-k^2 \sin^2 \varphi}} d\varphi > \frac{\pi}{2}. \quad (4.13)$$

On the other hand, for $u \geq \pi/2$ we have $E(u, k) \geq E(k) > 1$. Consequently,

$$\frac{F(u, k)}{1 + E(u, k)F(u, k)} < \frac{F(u, k)}{1 + F(u, k)} < 1. \quad (4.14)$$

Inequalities (4.13) and (4.14) contradict one to another. This completes the proof of this lemma. \square

THEOREM 4.5. *The first conjugate time for $\lambda \in C_1$ is bounded as $4K(k) \leq t_1^{\text{conj}}(\lambda) \leq 2p_1^1(k)$. Moreover,*

$$\begin{aligned} \lim_{k \rightarrow 0^+} t_1^{\text{conj}}(\lambda) &= 2\pi, \\ \lim_{k \rightarrow 1-0} t_1^{\text{conj}}(\lambda) &= +\infty. \end{aligned}$$

Proof. We first prove the lower bound of $t_1^{\text{conj}}(\lambda)$. We employ the approach adopted in the proof of Theorems 2.1, 2.2 [19] and prove that for $\lambda \in C_1$ the interval $(0, 2K(k))$ does not contain conjugate points for the extremal trajectory $q(t) = \text{Exp}(\lambda, t)$.

Given any $\hat{\lambda} \in C_1$, denote the corresponding elliptic coordinates $(\hat{\varphi}, \hat{k})$ and for $\hat{t} = 4K(\hat{k})$ denote the corresponding parameters (2.17) as $\hat{p} = \hat{t}/2$ and $\hat{\tau} = \hat{\varphi} + \hat{p}$. From the discussion on conjugate points it is clear that for $p \in (0, \hat{p})$, the extremal trajectory $\hat{q}(t) = \text{Exp}(\hat{\lambda}, t)$ does not have conjugate points if $J_1 \neq 0$.

We choose the following family of curves in the plane (k, p) continuous in the parameter s :

$$\{(k^s, p^s) \mid s \in [0, 1]\}, \quad k^s = s\hat{k}, \quad p^s = 2K(k^s). \quad (4.15)$$

Clearly the endpoints of the curve (k^s, p^s) are $(k^0, p^0) = (0, \pi)$ and $(k^1, p^1) = (\hat{k}, 2K(\hat{k}))$. The corresponding family of extremal trajectories is given as:

$$q^s(t) = \text{Exp}(\varphi^s, k^s, t), \quad t \in [0, t^s], \quad s \in [0, 1], \quad (4.16)$$

$$t^s = 2p^s, \quad \varphi^s = \hat{\tau} - p^s. \quad (4.17)$$

From Lemma 4.2 it is clear that for sufficiently small $s > 0$, the Jacobian $J > 0$ and hence the extremal trajectory $q^s(t)$ does not contain conjugate points for $p \in (0, 2K(k^s))$, i.e., for $t \in (0, 4K(k^s))$. Then from Proposition 4.1 it follows that the extremal trajectory $q^s(t)$ does not contain conjugate points for any $s \in [0, 1]$. Hence the extremal trajectory $q(t) = \text{Exp}(\lambda, t)$, $\lambda \in C_1$, does not contain conjugate points in the interval $(0, 4K(k))$ and therefore $t_1^{\text{conj}}(\lambda) \geq 4K(k)$.

For proof of upper bound apply Lemma 4.4. Hence it is proved that the first conjugate time is bounded as:

$$4K(k) \leq t_1^{\text{conj}}(\lambda) \leq 2p_1^1(k). \quad (4.18)$$

From Lemma 4.2 and (4.7), the first root of J occurs at $p = \pi$ and $\lim_{k \rightarrow 0^+} 2K(k) = \pi$. Therefore,

$$\lim_{k \rightarrow 0^+} t_1^{\text{conj}}(\lambda) = 4K(0) = 2\pi.$$

It can be readily seen that:

$$\lim_{k \rightarrow 1-0} t_1^{\text{conj}}(\lambda) = +\infty.$$

\square

REMARK 1. *For $\lambda \in C_1$, the instant $t = 4K(k)$ is conjugate iff $\text{sn}\tau = 0$. For proof substitute $n = 1$ in (4.9) Lemma 4.3 or alternatively substitute $\text{sn}\tau = 0$ in (4.6).*

4.3. Bounds for $t_1^{\text{conj}}(\lambda)$ for $\lambda \in C_2$. Using the elliptic coordinates (ψ, k) defined in Section 5.3.1 [6] and the parametrization of extremal trajectories (2.8), the Jacobian of the exponential mapping is given as:

$$J = \frac{\partial(x_t, y_t, z_t)}{\partial(\psi, k, t)} = \frac{-kJ_1(p, \tau, k)}{(1 - k^2)^2(1 - k \operatorname{sn} p \operatorname{sn} \tau)^2}, \quad (4.19)$$

where p and τ for $\lambda \in C_2$ were defined in (2.18) and J_1 is given by (4.6).

REMARK 2. Notice that the Jacobian for $\lambda \in C_2$ (4.19) is just $(-k)$ times the expression of Jacobian for $\lambda \in C_1$ (4.5). Such a symmetry is unexpected and was not observed in similar problems [18],[19],[20].

THEOREM 4.6. The first conjugate time for $\lambda \in C_2$ is bounded as $4kK(k) \leq t_1^{\text{conj}}(\lambda) \leq 2kp_1^1(k)$. Moreover,

$$\begin{aligned} \lim_{k \rightarrow 0} t_1^{\text{conj}}(\lambda) &= 0, \\ \lim_{k \rightarrow 1-0} t_1^{\text{conj}}(\lambda) &= +\infty. \end{aligned}$$

Proof. Since $J = -kJ_1$ for $\lambda \in C_2$, therefore all arguments presented in the proof of Theorem 4.5 apply. \square

REMARK 3. For $\lambda \in C_2$, the instant $t = 4kK(k)$ is conjugate iff $\operatorname{sn} \tau = 0$. For proof substitute $n = 1$ in (4.9) Lemma 4.3 or alternatively substitute $\operatorname{sn} \tau = 0$ in (4.6).

4.4. Conjugate Points for the Cases of Critical Energy of Pendulum. THEOREM 4.7. An extremal trajectory $q(t) = \operatorname{Exp}(\lambda, t)$ does not have any conjugate points for $t > 0$ when $\lambda \in C_3 \cup C_4 \cup C_5$.

Proof. Consider first the case when $\lambda \in C_4$. Since projections of extremal trajectories to xy -plane are straight lines given as $x_t = \operatorname{sgn}(\cos \frac{\gamma}{2})t$, $y_t = 0$ [6], thus they are globally optimal. Hence, there are no conjugate points for $\lambda \in C_4$.

Consider now the cases $\lambda \in C_3 \cup C_5$. From (2.8)–(2.12) and Figure 2.1 we see that $\lambda \in C_3 \cup C_5$ are the limiting cases for $\lambda \in C_1$. Thus conjugate points in this case can be calculated by limit passage $\lim_{k \rightarrow 1-0} t_1^{\text{conj}}(\lambda)$, $\lambda \in C_1$. From Theorem 4.5 we have:

$$\lim_{k \rightarrow 1-0} t_1^{\text{conj}}(\lambda) = +\infty. \quad (4.20)$$

Hence there are no conjugate points for $\lambda \in C_3 \cup C_5$. \square

THEOREM 4.8. The two sided bounds on $t_1^{\text{conj}}(\lambda)$ for $\lambda \in C_1$ given by Theorem 4.5 are exact in the following sense:

$$(1) \quad \text{If } \operatorname{sn} \tau = 0 \quad \text{then} \quad t_1^{\text{conj}}(\lambda) = 4K(k), \quad (4.21)$$

$$(2) \quad \text{If } \operatorname{cn} \tau = 0 \quad \text{then} \quad t_1^{\text{conj}}(\lambda) = p_1^1(k). \quad (4.22)$$

Proof. Substitute $\operatorname{sn} \tau = 0$ for item (1) and $\operatorname{cn} \tau = 0$ for item (2) in (4.6) respectively. \square

THEOREM 4.9. The two sided bounds on $t_1^{\text{conj}}(\lambda)$ for $\lambda \in C_2$ given by Theorem 4.6 are exact in the following sense:

$$(1) \quad \text{If } \operatorname{sn} \tau = 0 \quad \text{then} \quad t_1^{\text{conj}}(\lambda) = 4kK(k), \quad (4.23)$$

$$(2) \quad \text{If } \operatorname{cn} \tau = 0 \quad \text{then} \quad t_1^{\text{conj}}(\lambda) = 2kp_1^1(k). \quad (4.24)$$

Proof. Substitute $\operatorname{sn} \tau = 0$ for item (1) and $\operatorname{cn} \tau = 0$ for item (2) in (4.6) respectively. \square

4.5. n -th Conjugate Time . Computation of the first conjugate time is important in the study of local optimality of the extremal trajectories. It turns out that in the study of sub-Riemannian wavefront, it is essential to bound not only the first conjugate time, but all other conjugate times as well. Hence in the following, we obtain the bounds for the n -th conjugate time $t_n^{\text{conj}}(\lambda)$ for $\lambda \in C_1 \cup C_2$. Recall that if $\lambda \in C_3 \cup C_4 \cup C_5$, then the trajectory $\operatorname{Exp}(\lambda, t)$ is free of conjugate points (Theorem 4.7).

THEOREM 4.10. *The n -th conjugate time $t_n^{\text{conj}}(\lambda)$ for $\lambda \in C_1$ is bounded as $4nK(k) \leq t_{2n-1}^{\text{conj}}(\lambda) \leq 2p_1^n(k)$ and $2p_1^n(k) \leq t_{2n}^{\text{conj}}(\lambda) \leq 4(n+1)K(k)$, $\forall n \in \mathbb{N}$.*

Proof. From Lemma 4.3 it is readily seen that $\forall p = 2nK(k)$, the expression of the Jacobian $J_1 \geq 0$. From Lemma 4.4, $J_1 \leq 0$ at $p = p_1^n(k)$ i.e., the n -th root of the function $f_1(p)$. Hence Jacobian J (4.5) takes values of opposite signs at the endpoints of the intervals $[2nK(k), p_1^n(k)]$ and $[p_1^n(k), 2(n+1)K(k)]$. Therefore, the n -th conjugate time $t_n^{\text{conj}}(\lambda)$ is bounded as $4nK(k) \leq t_{2n-1}^{\text{conj}}(\lambda) \leq 2p_1^n(k)$ and $2p_1^n(k) \leq t_{2n}^{\text{conj}}(\lambda) \leq 4(n+1)K(k)$ $\forall n \in \mathbb{N}$. \square

COROLLARY 4.11. *From Theorem 4.6 and Theorem 4.10 we see that the n -th conjugate time $t_n^{\text{conj}}(\lambda)$ for $\lambda \in C_2$ is bounded as $4nkK(k) \leq t_{2n-1}^{\text{conj}}(\lambda) \leq 2kp_1^n(k)$ and $2kp_1^n(k) \leq t_{2n}^{\text{conj}}(\lambda) \leq 4(n+1)kK(k)$.*

THEOREM 4.12. *The n -th conjugate times are bounded as:*

$$\begin{aligned} \lambda \in C_1 &\implies 4nK(k) \leq t_{2n-1}^{\text{conj}}(\lambda) \leq 2p_1^n(k), \quad 2p_1^n(k) \leq t_{2n}^{\text{conj}}(\lambda) \leq 4(n+1)K(k), \\ \lambda \in C_2 &\implies 4nkK(k) \leq t_{2n-1}^{\text{conj}}(\lambda) \leq 2kp_1^n(k), \quad 2kp_1^n(k) \leq t_{2n}^{\text{conj}}(\lambda) \leq 4(n+1)kK(k). \end{aligned}$$

Proof. The bounds follow from Theorem 4.10 and Corollary 4.11. \square

REMARK 4. *Notice that any extremal trajectory $q(t)$ either has countable number of conjugate points, or is free of conjugate points. This alternative is similar to that for LQ problems [22].*

4.6. Sub-Riemannian Sphere and Wavefront. Having explicit parametrization of the exponential mapping $\text{Exp}(\lambda, t)$, $\lambda \in C$, $t > 0$ and the global bound on the cut time, we perform a graphic study of some essential objects in the sub-Riemannian problem on $\text{SH}(2)$ in the rectifying coordinates (R_1, R_2, z) . In particular we plot the sub-Riemannian sphere S_R and the sub-Riemannian wavefront W_R . Recall that the sub-Riemannian wavefront $W_R(q_0; R)$ at q_0 is the set of end-points of geodesics with sub-Riemannian length R starting from q_0 and the sub-Riemannian sphere $S_R(q_0; R)$ at q_0 is the set of end-points of minimizing geodesics of sub-Riemannian length R and starting from q_0 :

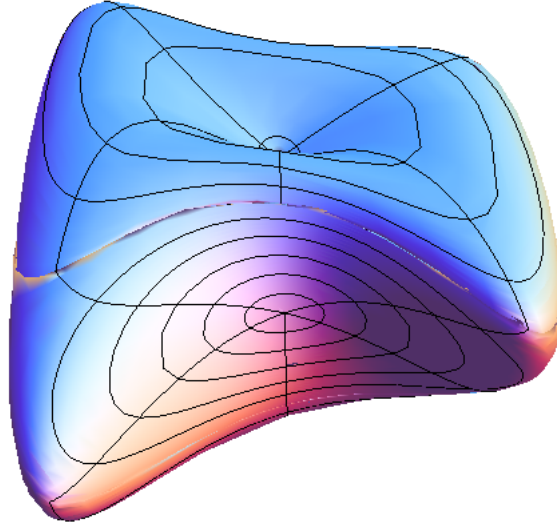
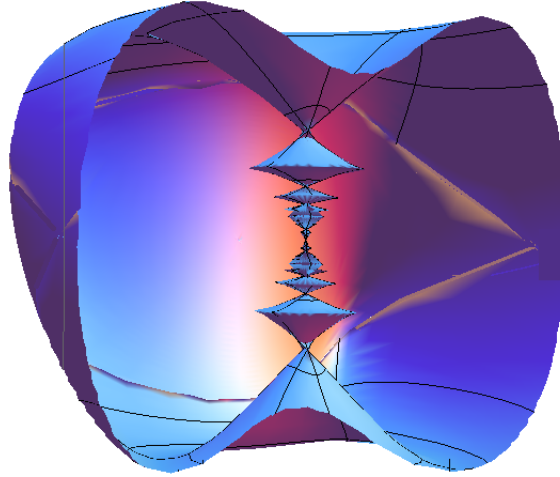
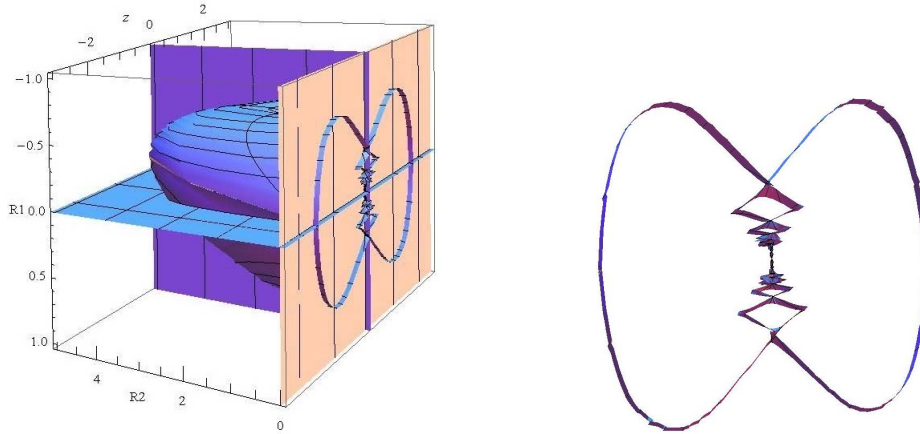
$$\begin{aligned} W_R &= \{q = \text{Exp}(\lambda, R) \in M \mid \lambda \in C\}, \\ S_R &= \{q = \text{Exp}(\lambda, R) \in M \mid \lambda \in C, \quad t_{\text{cut}}(\lambda) \geq R\} = \{q \in M \mid d(q_0, q) = R\}, \end{aligned}$$

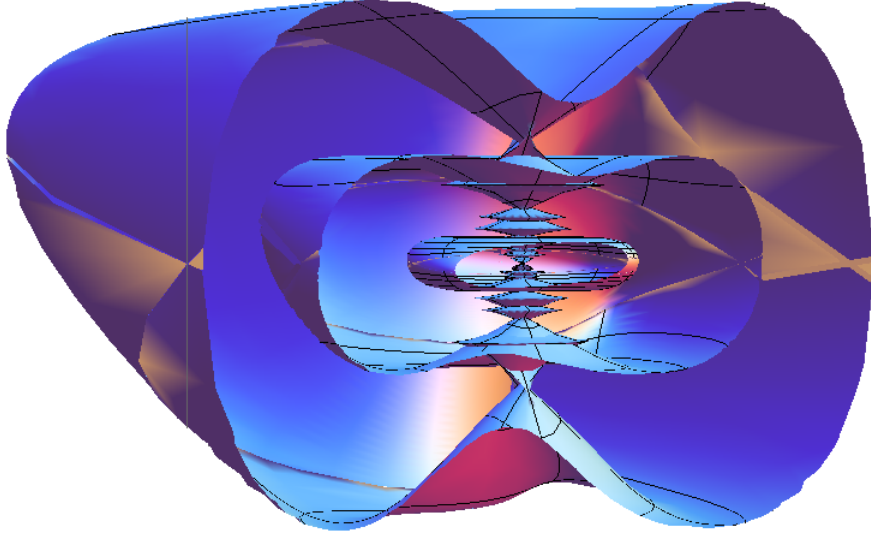
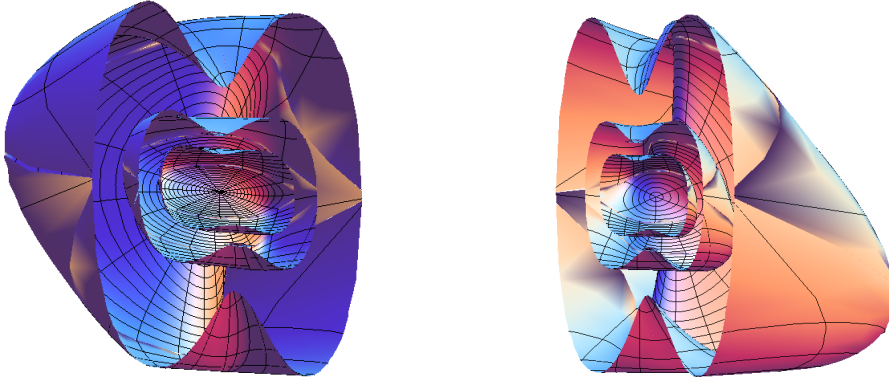
where R is the radius of sub-Riemannian sphere or wavefront and $d(q_0, q_1) = \inf\{l(q(\cdot))\}$ is the sub-Riemannian distance corresponding to sub-Riemannian length functional $l(q(\cdot))$ (2.4). Note the essential difference between sub-Riemannian wavefront and sub-Riemannian sphere. The geodesics in sub-Riemannian wavefront are only locally minimizing and drawn for time greater than the cut time as well. On the contrary, the geodesics in sub-Riemannian sphere are globally minimizing and therefore drawn for time not greater than the upper bound of cut time and therefore, $S_R \subset W_R$, but $S_R \neq W_R$ for $R > 0$ and S_R is the exterior component of W_R in the following sense:

$$S_R = \partial(M \setminus W_R).$$

A plot of the sub-Riemannian sphere is presented in Figure 4.4 and plots of cutout of the sub-Riemannian wavefront are presented in Figures 4.5–4.6. From Figure 4.6 it is clear that the wavefront has self intersections in the surfaces $R_i(q_t) = 0$ and $z_t = 0$ as expected from the general and complete description of Maxwell strata. Figure 4.7 shows the Matryoshka of the sub-Riemannian wavefront where self intersections in wavefronts of different radii are clearly visible. In Figure 4.8 we present the Matryoshka of the sub-Riemannian spheres S_R for different $R > 0$. Plots are presented from two different viewpoints for better visualization. Note that as expected, exterior view of the sub-Riemannian sphere is same as that of wavefront.

5. Future Work. In this paper we extended our research on the sub-Riemannian problem on the Lie group $\text{SH}(2)$ that was initiated in [6]. We obtained complete description of the Maxwell points, calculated the upper bound on the cut time and computed the exact upper and lower bounds for the n -th conjugate time, $n \in \mathbb{N}$. The next research direction is the global optimality of sub-Riemannian geodesics. In this regard we conjecture that the cut time is equal to the first Maxwell time corresponding to the group of discrete symmetries of the exponential mapping. This conjecture will be the subject of our forthcoming work on the sub-Riemannian problem on $\text{SH}(2)$.

FIG. 4.4. *Sub-Riemannian sphere for $R = 2$* FIG. 4.5. *Cutout of the sub-Riemannian wavefront for $R = 2$* FIG. 4.6. *Cutout of the sub-Riemannian wavefront with self intersections in the planes $R_i(q_t) = 0$ and $z_t = 0$ for $R = 2$*

FIG. 4.7. Matryoshka of sub-Riemannian wavefronts W_R for $R = 1, 2, 3$ FIG. 4.8. Matryoshka of sub-Riemannian spheres S_R for $R = 1, 2, 3$

6. Conclusion. The study of the sub-Riemannian problem on the group $SH(2)$ is an important research goal that was initiated in [6] and has been continued in this work. We obtained a complete description of the Maxwell points and global upper bound on the cut time. We also computed the exact lower and upper bound of the n -th conjugate time. We discovered an unexpected symmetry in the Jacobian expression and the conjugate points in the case of oscillating and rotating pendulum which hasn't been observed in optimality analysis in sub-Riemannian problem on $SE(2)$ [19], the Engel group [20] and the Euler elastic problem [18]. We conclude that the n -th conjugate time is bounded by similar functions from below and above for both $\lambda \in C_1$ and $\lambda \in C_2$. Moreover, we showed that each geodesic contains either zero or a countable number of conjugate points. **Acknowledgment** We thank Prof. A. Yu. Popov for the proof of Lemma 4.4.

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