

Symmetries and Maxwell points in the plate-ball problem

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The plate-ball problem

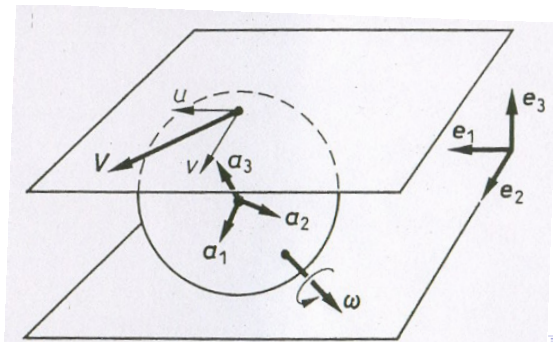
Rolling of sphere on plane without slipping or twisting

Given: $A, B \in \mathbb{R}^2$, frames (a_1, a_2, a_3) , (b_1, b_2, b_3) in \mathbb{R}^3 .

Find: $\gamma(t) \in \mathbb{R}^2$, $t \in [0, t_1]$, — the shortest curve s.t.:

$\gamma(0) = A$, $\gamma(t_1) = B$,

by rolling along $\gamma(t)$, orientation of the sphere transfers from (a_1, a_2, a_3) to (b_1, b_2, b_3) .



State and control variables

- Contact point $(x, y) \in \mathbb{R}^2$
- Orientation of sphere $R : a_i \mapsto e_i, i = 1, 2, 3, \quad R \in \text{SO}(3)$
- State of the system $Q = (x, y, R) \in \mathbb{R}^2 \times \text{SO}(3) = M$
- Boundary conditions $Q(0) = Q_0, Q(t_1) = Q_1$
- Controls $u_1 = u/2, u_2 = v/2$

- Cost functional

$$I(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min$$

Control system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad (x, y) \in \mathbb{R}^2, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$\dot{R} = R\Omega, \quad R \in SO(3), \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \text{ angular velocity vector.}$$

$$\text{No twisting} \quad \Rightarrow \quad \omega_3 = 0.$$

$$\text{No slipping} \quad \Rightarrow \quad \omega_1 = u_2, \quad \omega_2 = -u_1.$$

$$\Omega = \begin{pmatrix} 0 & 0 & -u_1 \\ 0 & 0 & -u_2 \\ u_1 & u_2 & 0 \end{pmatrix}$$

History of the problem

1894 H. Hertz: rolling sphere as a nonholonomic mechanical system.

1983 J.M. Hammersley: statement of the plate-ball problem.

1986 A.M. Arthur, G.R. Walsh: integrability of Hamiltonian system of PMP in quadratures.

1990 Z. Li, E. Canny: complete controllability of the control system.

1993 V. Jurdjevic:

- projections of extremal curves $(x(t), y(t))$ — Euler elasticae,
- description of qualitative types of extremal trajectories,
- quadratures for evolution of Euler angles along extremal trajectories.

New results

- Continuous and discrete symmetries
- Fixed points of symmetries (Maxwell points)
- Necessary optimality conditions
- Asymptotics of extremal trajectories and limit behavior of Maxwell points for sphere rolling on sinusoids of small amplitude
- Global structure of the exponential mapping

Existence of solutions

- Left-invariant sub-Riemannian problem:

$$\begin{aligned}\dot{Q} &= u_1 X_1(Q) + u_2 X_2(Q), & (u_1, u_2) &\in \mathbb{R}^2, \\ Q(0) &= Q_0, & Q(t_1) &= Q_1, & Q &\in M = \mathbb{R}^2 \times \text{SO}(3), \\ I &= \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min.\end{aligned}$$

- Complete controllability by Rashevskii-Chow theorem:

$$\begin{aligned}\text{span}_Q(X_1, X_2, X_3, X_4, X_5) &= T_Q M \quad \forall Q \in M, \\ X_3 &= [X_1, X_2], & X_4 &= [X_1, X_3], & X_5 &= [X_2, X_3].\end{aligned}$$

- Filippov's theorem: $\forall Q_0, Q_1 \in M$ optimal trajectory exists.
- $Q_0 = (0, 0, \text{Id}) \in \mathbb{R}^2 \times \text{SO}(3)$.

Pontryagin maximum principle

- Abnormal extremal trajectories: rolling of sphere along straight lines.
- Normal extremals:

$$\dot{\theta} = c, \quad \dot{c} = -r \sin \theta, \quad \dot{\alpha} = \dot{r} = 0, \quad (1)$$

$$\dot{x} = \cos(\theta + \alpha), \quad \dot{y} = \sin(\theta + \alpha), \quad (2)$$

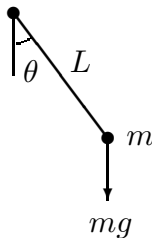
$$\dot{R} = R(\sin(\theta + \alpha)A_1 - \cos(\theta + \alpha)A_2), \quad (3)$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = [A_1, A_2] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

- (1) mathematical pendulum,
(2) Euler elasticae.

Mathematical pendulum $\dot{\theta} = c, \dot{c} = -r \sin \theta$



- $\lambda = (\theta, c, r) \in C = \{\theta \in S^1, c \in \mathbb{R}, r \geq 0\}$,
- Energy $E = c^2/2 - r \cos \theta = \text{const} \in [-r, +\infty)$,
- $r = g/L \geq 0$.

Decomposition of the phase cylinder C of pendulum

$$C = \bigcup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j$$

$$C_1 = \{\lambda \in C \mid E \in (-r, r), r > 0\},$$

$$C_2 = \{\lambda \in C \mid E \in (r, +\infty), r > 0\},$$

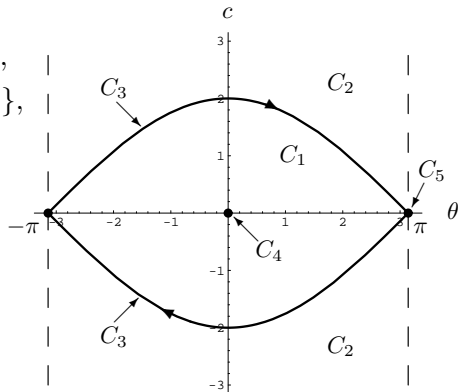
$$C_3 = \{\lambda \in C \mid E = r > 0, c \neq 0\},$$

$$C_4 = \{\lambda \in C \mid E = -r, r > 0\},$$

$$C_5 = \{\lambda \in C \mid E = r > 0, c = 0\},$$

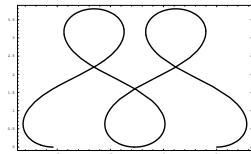
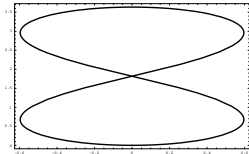
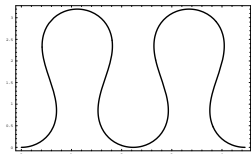
$$C_6 = \{\lambda \in C \mid r = 0, c \neq 0\},$$

$$C_7 = \{\lambda \in C \mid r = 0, c = 0\}.$$

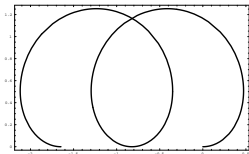


Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

C_1 : inflectional elasticae

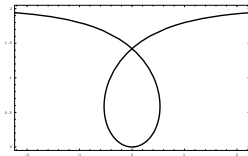


C_2 : non-inflectional elasticae



Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

C_3 : critical elasticae



C_4 , C_5 , C_7 : straight lines

C_6 : circles

Integration of normal Hamiltonian system of PMP

$$\theta = c, \quad \dot{c} = -r \sin \theta, \quad \dot{x} = \cos(\theta + \alpha), \quad \dot{y} = \sin(\theta + \alpha), \\ \dot{R} = R(\sin(\theta + \alpha)A_1 - \cos(\theta + \alpha)A_2).$$

- $\theta(t), c(t), x(t), y(t)$: Jacobi's functions $\text{cn}, \text{sn}, \text{dn}, E$,

$$\text{cn}(u, k) = \cos(\text{am}(u, k)),$$

$$\varphi = \text{am}(u, k) \iff u = \int_0^\varphi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = F(\varphi, k).$$

- $R(t) = e^{(\alpha - \varphi_3^0)A_3} e^{-\varphi_2^0 A_2} e^{\varphi_1(t)A_3} e^{\varphi_2(t)A_2} e^{(\varphi_3(t) - \alpha)A_3}$,
 $\varphi_i(t)$: Jacobi's functions + elliptic integral of the 3-rd kind

$$\Pi(n, u, k) = \int_0^u \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - k^2 \sin^2 t}}.$$

Optimality of extremal trajectories

- Short arcs of extremal trajectories $Q(s)$ are optimal
- Cut time along $Q(s)$:

$$t_{\text{cut}} = \sup\{t > 0 \mid Q(s), s \in [0, t], \text{ is optimal } \}.$$

- Maxwell time:

$$\exists \tilde{Q}(s) \neq Q(s), \quad Q(0) = \tilde{Q}(0) = Q_0,$$

$$Q(t) = \tilde{Q}(t) \text{ Maxwell point,}$$

$$t = t_{\text{Max}} \text{ Maxwell time.}$$

- Upper bound on cut time:

$$t_{\text{cut}} \leq t_{\text{Max}}.$$

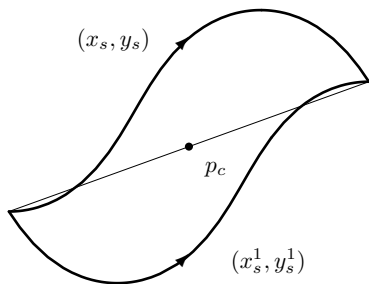
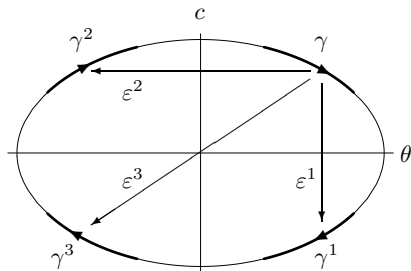
Rotations Φ^β , $\beta \in S^1$

$$(\theta, c, r, \alpha) \mapsto (\theta, c, r, \alpha + \beta),$$

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} \mapsto \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix},$$

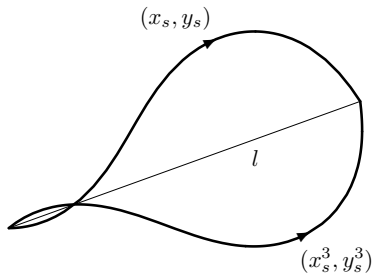
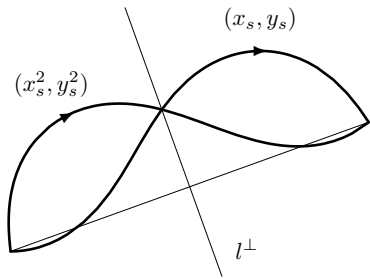
$$R_s \mapsto e^{\beta A_3} R_s e^{-\beta A_3}.$$

Reflections ε^i



$$\begin{aligned}\varepsilon^1: (\theta_s, c_s) &\mapsto (\theta_{t-s}, -c_{t-s}), s \in [0, t] \\ (x_s, y_s) &\mapsto (x_s^1, y_s^1) = (x_{t-s} - x_t, y_{t-s} - y_t) \\ R_s &\mapsto (R_t)^{-1} R_{t-s}\end{aligned}$$

Reflections ε^i



$$\begin{aligned} \varepsilon^2: (\theta_s, c_s) &\mapsto (-\theta_{t-s}, c_{t-s}), s \in [0, t] \\ (x_s, y_s) &\mapsto (x_s^2, y_s^2) = (x_{t-s} - x_t, y_t - y_{t-s}) \\ R_s &\mapsto l_2(R_t)^{-1}R_{t-s}l_2, l_2 = e^{\pi A_2}. \end{aligned}$$

$$\begin{aligned} \varepsilon^3: (\theta_s, c_s) &\mapsto (-\theta_s, -c_s), s \in [0, t] \\ (x_s, y_s) &\mapsto (x_s^3, y_s^3) = (x_s, -y_s) \\ R_s &\mapsto l_2R_sl_2. \end{aligned}$$

Exponential mapping and its symmetries

- Group of symmetries

$$G = \langle \Phi^\beta, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle = \{ \Phi^\beta, \Phi^\beta \circ \varepsilon^i \mid \beta \in S^1, i = 1, 2, 3 \}$$

- Exponential mapping

$$\begin{aligned} \text{Exp}(\lambda, s) = Q_s &= (x_s, y_s, R_s) \in M = \mathbb{R}^2 \times \text{SO}(3), \\ \lambda &= (\theta, c, \alpha, r) \in C, \quad s > 0. \end{aligned}$$

- Symmetries of exponential mapping

$$\begin{array}{ccc} C \times \mathbb{R}_+ & \xrightarrow{\text{Exp}} & M \\ \downarrow \varepsilon^i \circ \Phi^\beta & & \downarrow \varepsilon^i \circ \Phi^\beta \\ C \times \mathbb{R}_+ & \xrightarrow{\text{Exp}} & M \end{array}$$

$$\begin{array}{ccc} (\lambda, t) & \xrightarrow{\text{Exp}} & Q_t \\ \downarrow \varepsilon^i \circ \Phi^\beta & & \downarrow \varepsilon^i \circ \Phi^\beta \\ (\lambda^{i,\beta}, t) & \xrightarrow{\text{Exp}} & Q_t^{i,\beta} \end{array}$$

Maxwell sets corresponding to reflections

- $MAX^i = \{(\lambda, t) \mid \exists \beta \in S^1 : \lambda^{i,\beta} \neq \lambda, Q_t = Q_t^{i,\beta}\},$
 $i = 1, 2, 3.$

- Necessary optimality conditions:

$$(\lambda, t) \in MAX^i \quad \Rightarrow \quad Q_s = \text{Exp}(\lambda, s) \text{ not optimal for } s > t, \\ t_{\text{cut}}(\lambda) \leq t.$$

Representation of rotations in \mathbb{R}^3 by quaternions

- $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, \dots, q_3 \in \mathbb{R}\}$
- $S^3 = \{q \in \mathbb{H} \mid |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$
- $I = \{q \in \mathbb{H} \mid \operatorname{Re} q = q_0 = 0\}$
- $q \in S^3 \Rightarrow R_q(a) = qaq^{-1}, \quad a \in I, \quad R_q \in \operatorname{SO}(3) \cong \operatorname{SO}(I)$
- lift of the system $\dot{R} = R\Omega$ from $\operatorname{SO}(3)$ to S^3 :

$$\begin{cases} \dot{q}_0 = \frac{1}{2}(q_2 u_1 - q_1 u_2), \\ \dot{q}_1 = \frac{1}{2}(q_3 u_1 + q_0 u_2), \\ \dot{q}_2 = \frac{1}{2}(-q_0 u_1 + q_3 u_2), \\ \dot{q}_3 = \frac{1}{2}(-q_1 u_1 - q_2 u_2), \end{cases} \quad q \in S^3, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$q(0) = 1.$

Necessary optimality conditions in terms of MAX^1

Theorem

Let $Q_s = (x_s, y_s, R_s) = \text{Exp}(\lambda, s)$, $t > 0$ satisfy the conditions:

- (1) $q_3(t) = 0$,
- (2) *elastica* $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate and not centered at inflection point.

Then $(\lambda, t) \in \text{MAX}^1$, thus for any $t_1 > t$ the trajectory Q_s , $s \in [0, t_1]$, is not optimal.

$$q_3(t) = 0 \iff \text{axis of rotation } (q_1(t), q_2(t), q_3(t)) \parallel \mathbb{R}^2$$

Necessary optimality conditions in terms of MAX^2

Theorem

Let $Q_s = (x_s, y_s, R_s) = \text{Exp}(\lambda, s)$, $t > 0$ satisfy the conditions:

- (1) $(xq_1 + yq_2)(t) = 0$,
- (2) *elastica* $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate and not centered at vertex.

Then $(\lambda, t) \in \text{MAX}^2$, thus for any $t_1 > t$ the trajectory Q_s , $s \in [0, t_1]$, is not optimal.

$$(xq_1 + yq_2)(t) = 0 \iff (q_1(t), q_2(t), q_3(t)) \perp (x(t), y(t), 0)$$

Asymptotics of extremal trajectories near $(\theta, c) = (0, 0)$

- $m = \sqrt{r}$, $s = mt$, $d = c/m$,
$$\begin{pmatrix} x \\ y \end{pmatrix} = A(\alpha) \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A(\alpha) \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}$$
$$A(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad q_0 = \bar{q}_0, \quad q_3 = \bar{q}_3.$$

- $\rho_0 = \sqrt{\theta_0^2 + d_0^2} \rightarrow 0.$

- oscillations of pendulum $\theta' = d$, $d' = -\sin \theta$:

$$\theta(s) = \theta_0 \cos s + d_0 \sin s + O(\rho_0^2).$$

- elasticae \approx sinusoids of small amplitude:

$$\bar{x}(s) = \frac{s}{m} + O(\rho_0^2),$$

$$\bar{y}(s) = \frac{1}{m}(\theta_0 \sin s + d_0(1 - \cos s)) + O(\rho_0^2).$$

Asymptotics of extremal trajectories near $(\theta, c) = (0, 0)$

$$\bar{q}_0(s) = \cos \frac{s}{2m} + O(\rho_0^2),$$

$$\begin{aligned}\bar{q}_1(s) = & \frac{1}{2(m^2 - 1)} \left(m \cos \frac{s}{2m} \sin s - (1 + \cos s) \sin \frac{s}{2m} \right) \theta_0 + \\ & + \frac{1}{2(m^2 - 1)} \left(m(1 - \cos s) \cos \frac{s}{2m} - \sin s \sin \frac{s}{2m} \right) d_0 + O(\rho_0^2),\end{aligned}$$

$$\bar{q}_2(s) = -\sin \frac{s}{2m} + O(\rho_0^2),$$

$$\begin{aligned}\bar{q}_3(s) = & \frac{1}{2(m^2 - 1)} \left((-1 + \cos s) \cos \frac{s}{2m} + m \sin s \sin \frac{s}{2m} \right) \theta_0 + \\ & + \frac{1}{2(m^2 - 1)} \left(\sin s \cos \frac{s}{2m} - m(1 + \cos s) \sin \frac{s}{2m} \right) d_0 + O(\rho_0^2).\end{aligned}$$

Limit behavior of Maxwell set MAX^1 as $\rho_0 \rightarrow 0$

- Th.: $q_3(t) = 0$, elastica not centered at inflection point
 $\Rightarrow (\lambda, t) \in \text{MAX}^1$.
- $q_3 = \frac{d_0 \cos p - \theta_0 \sin p}{m^2 - 1} \cdot g_1(p, m) + O(\rho_0^2)$, $p = mt/2$.
- $g_1(p, m) = \cos \frac{p}{m} \sin p - m \cos p \sin \frac{p}{m}$
- $p_1(m) = \min\{p > 0 \mid g_1(p, m) = 0\}$
- $Q_t = \text{Exp}(\lambda, t)$, $\lambda = (\theta_0, d_0, m, \alpha) \in C_1$,
 $\rho_0 \rightarrow 0$, $m \rightarrow \bar{m} \Rightarrow$

$\forall \varepsilon > 0 \exists t \in (t_1 + \varepsilon, t_1 + \varepsilon)$ such that $(\lambda, t) \in \text{MAX}^1$,
trajectory Q_t not optimal at $t \in [0, t_1 + \varepsilon]$, $t_1 = 2p_1(\bar{m})/\bar{m}$.

Bounds of the first root of equation $g_1(\rho, m) = 0$

Theorem

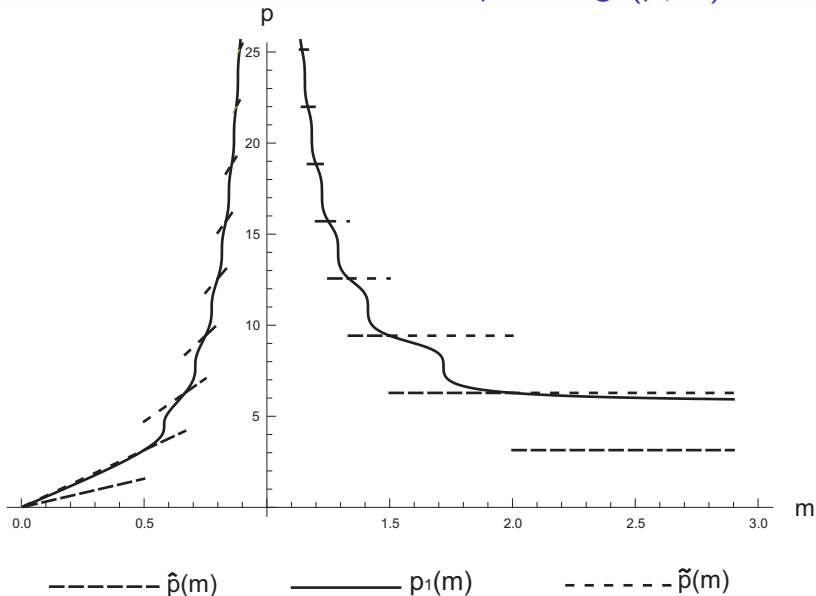
For any $m \in (0, 1) \cup (1, +\infty)$,

$$\hat{p}(m) \leq \rho_1(m) < \tilde{p}(m),$$

where

$$\hat{p}(m) = \begin{cases} m\rho, & \text{for } m \in (0, \frac{1}{2}], \\ m\pi([\frac{m}{1-m}] + 1), & \text{for } m \in (\frac{1}{2}, 1), \\ \pi([\frac{1}{m-1}] + 1), & \text{for } m \in (1, 2], \\ \rho, & \text{for } m > 2, \quad \rho = \tan \rho, \quad \rho \in (\pi, \frac{3\pi}{2}) \end{cases}$$
$$\tilde{p}(m) = \begin{cases} m\pi([\frac{m}{1-m}] + 2), & \text{for } m \in (0, 1), \\ \pi([\frac{1}{m-1}] + 2), & \text{for } m \in (1, +\infty). \end{cases}$$

Bounds of the first root of equation $g_1(p, m) = 0$



Properties of the first root of equation $g_1(p, m) = 0$

Theorem

- a) $\lim_{m \rightarrow 1} p_1(m) = +\infty$;
- b) $p_1(m)$ grows for $m \in (0, 1)$ and decreases for $m \in (1, +\infty)$;
- c) $p_1(1 + \frac{1}{n}) = \pi(n + 1)$, $p_1(\frac{n}{n+1}) = \pi n$,
 $p_1(1 + \frac{2}{2n+1}) = \pi(n + \frac{3}{2})$, $p_1(\frac{2n+1}{2n+3}) = \pi(n + \frac{1}{2})$;
- d) $p_1(m)$ is continuous for $m \in (0, 1) \cup (1, +\infty)$;
- e) $p_1(m)$ is smooth for

$$m \in (0, 1) \cup (1, +\infty) \setminus (\{\frac{n}{n+1} | n \in \mathbb{N}\} \cup \{\frac{n+1}{n} | n \in \mathbb{N}\}).$$

$$\forall n \in \mathbb{N}: p_1'(\frac{n}{n+1}) = +\infty, p_1'(\frac{n+1}{n}) = -\infty.$$

Limit behavior of Maxwell set MAX^1

- $d = \rho \cos \chi, \theta = \rho \sin \chi$
- $\bar{\chi} \in S^1, \bar{m} > 0, \bar{m} \neq 1$
- $D_\delta = \{\lambda = (\rho, \chi, m, \alpha) \in C_1 \mid 0 < \rho < \delta, |\chi - \bar{\chi}| < \delta, |m - \bar{m}| < \delta\}$.
- $t_1 = t_1(\bar{m}) = 2p_1(\bar{m})/\bar{m}$
- $I_\varepsilon = \{t > 0 \mid t_1 - \varepsilon < t < t_1 + \varepsilon\}, \quad \varepsilon > 0.$

Theorem

Let $\bar{m} \neq 1, \cos(\bar{\chi} + t_1 \bar{m}/2) \neq 0$, then

$\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 \forall \lambda \in D_\delta \exists t \in I_\varepsilon$ such that $(\lambda, t) \in \text{MAX}^1$.

Bounds on cut time near $(\theta, c) = (0, 0)$

Theorem

Let $\lambda = (\rho, \chi, m, \alpha) \in C_1$ and $\bar{m} > 0$, $\bar{m} \neq 1$. Then

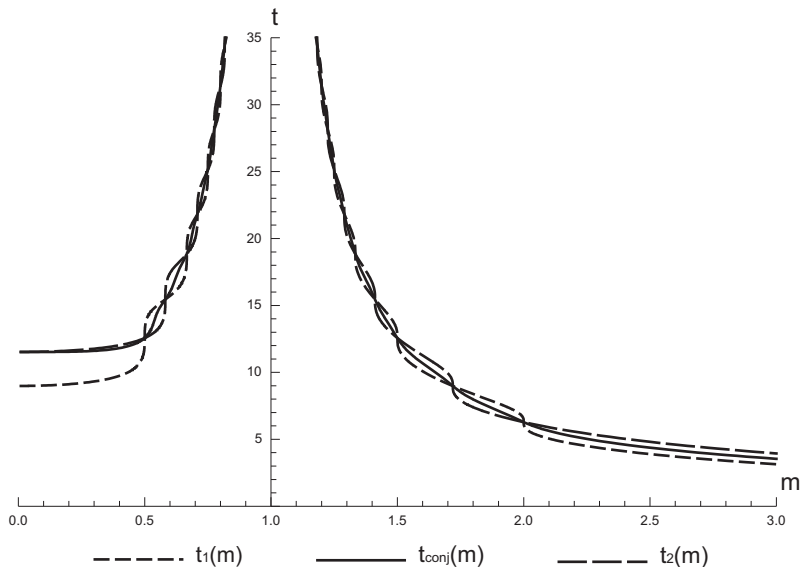
$$\overline{\lim}_{\rho \rightarrow 0, m \rightarrow \bar{m}} t_{\text{cut}}(\lambda) \leq t_1(\bar{m}).$$

Theorem

Let $\lambda = (\rho, \chi, m, \alpha) \in C_1$. Let $K \subset \{m \in \mathbb{R} \mid m > 0, m \neq 1\}$ be a compact. Then

$$\overline{\lim}_{\rho \rightarrow 0, m \in K} t_{\text{cut}}(\lambda) \leq \max_{m \in K} t_1(m).$$

Maxwell times and conjugate time as $(\theta, c) \rightarrow 0$



Global structure of exponential mapping

- $\text{Exp} : N \rightarrow M$,
 $N = C \times \mathbb{R}_+ = \{(\theta, c, \alpha, r, t) \mid \theta \in S^1, c \in \mathbb{R}, \alpha \in S^1, r \geq 0, t > 0\}$,
 $M = \mathbb{R}^2 \times \text{SO}(3)$
- $\forall Q_1 \in M \setminus Q_0 \exists (\lambda, t) \in N$ such that $Q_s = \text{Exp}(\lambda, s)$ optimal,
 $Q_1 = \text{Exp}(\lambda, t)$
 $t \leq t_{\text{cut}}(\lambda) \leq t_{\text{Max}}^1 = \inf\{s \mid (\lambda, s) \in \text{MAX}^1 \cup \text{MAX}^2\}$
 $(\lambda, t) \in \hat{N} = \{(\lambda, s) \in N \mid \lambda \in C, 0 < s \leq t_{\text{Max}}^1\}$
 $\text{Exp} : \hat{N} \rightarrow M \setminus Q_0$ surjective

Global structure of exponential mapping

- Decomposition in the preimage of Exp:

$$\widehat{N} \supset \bigcup_{i=1}^4 N_i, \text{cl}(\bigcup_{i=1}^4 N_i) \supset \widehat{N},$$

$$N_i = \{(\lambda, t) \in D_i \mid 0 < t < t_{\text{Max}}^1(\lambda)\},$$

$$D_i = \{(\lambda, t) \in N \mid \text{sgn } c_{t/2} = \pm 1, \text{sgn } \sin \theta_{t/2} = \pm 1\}.$$

- Decomposition in the image of Exp:

$$M \supset M_1 \cup M_2, \text{cl}(M_1 \cup M_2) = M,$$

$$M_i = \{(x, y, Q) \in M \mid q_3 > 0, \text{sgn}(xq_1 + yq_2) = \pm 1\}.$$

- Conjecture:

Exp : $N_1, N_3 \rightarrow M_1$ are diffeomorphisms,

Exp : $N_2, N_4 \rightarrow M_2$ are diffeomorphisms.

Steps required to prove the conjecture

- N_i, M_i connected, open (proved: diffeomorphic to $\mathbb{R}^4 \times S^1$),
- $N_i/\{\Phi^\beta \mid \beta \in S^1\}, M_i/\{\Phi^\beta \mid \beta \in S^1\}$ simply connected (proved : diffeomorphic to \mathbb{R}^4),
- $\text{Exp}(N_1), \text{Exp}(N_3) \subset M_1, \text{Exp}(N_2), \text{Exp}(N_4) \subset M_2$ (proved),
- $\text{Exp}(\partial N_i) \subset \partial M_1 \cup \partial M_2$ (proved),
- $\text{Exp} : N_1, N_3 \rightarrow M_1, \text{Exp} : N_2, N_4 \rightarrow M_2$ proper (partially proved),
- $\text{Exp}|_{N_i}$ nondegenerate (numerical evidence).

Algorithm for solution to the problem (modulo the conjecture)

- $Q_1 \in M_1 \cup M_2 \Rightarrow$ optimal trajectory $Q_s = ?$
- $Q_1 \in M_1 \Rightarrow$
 $\exists!(\lambda_1, t_1) \in N_1$ such that $\text{Exp}(\lambda_1, t_1) = Q_1$,
 $\exists!(\lambda_3, t_3) \in N_3$ such that $\text{Exp}(\lambda_3, t_3) = Q_1$,
 $t_1 < t_3 \Rightarrow Q_s^1 = \text{Exp}(\lambda_1, s)$ optimal,
 $t_1 > t_3 \Rightarrow Q_s^3 = \text{Exp}(\lambda_3, s)$ optimal,
 $t_1 = t_3 \Rightarrow Q_s^1, Q_s^3$ optimal.
- $Q_1 \in M_2 \Rightarrow$ similarly for $(\lambda_2, t_2) \in N_2, (\lambda_4, t_4) \in N_4$.

Results and plans

- parameterization of extremal trajectories,
- symmetries and Maxwell points,
- upper bound on cut time,
- asymptotic case $(\theta, c) \rightarrow (0, 0)$,
- global structure of the exponential mapping,
- software for numerical solution to the plate-ball problem (movies for MAX^i).