Optimal control problems on Lie groups related to Euler's elasticae (and pendulum)

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in collaboration with

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Summary of the talk

1. Symmetries and optimality
2. Euler's elastic problem
3. Sub-Riemannian problem on $\mathrm{E}(2)$
4. Nilpotent $(2, 3, 5)$ sub-Riemannian problem
5. The ball-plate problem

Left-invariant optimal control problems on Lie groups

$$\dot{q} = f(q,u), \qquad q \in M, \quad u \in U,$$
$$q(0) = q_0 = \mathrm{Id}, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed},$$
$$J_{t_1}[u] = \int_0^{t_1} g(u(t))\, dt \to \min .$$

- M Lie group, U smooth manifold,
- $f = qf(u)$, $g = g(u)$ left-invariant.

Suppose: $\forall\ q_1 \in \mathcal{A}_{q_0}(t_1)$, optimal control $u(\cdot) \in L_\infty$ exists.

PMP for left-invariant problems

- Trivialization $T^*M \cong L^* \times M$, $L = T_{\mathrm{Id}}M$ Lie algebra of M
- $F : L^* \times M \to T^*M$, $F(x, q) = \bar{x}_q$:
  $\langle \bar{x}_q, qa \rangle = \langle x, a \rangle$, $x \in L^*$, $a \in L$
- maximized normal Hamiltonian
  $H = \max_{u \in U} (\langle x, f(u) \rangle - g(u)) = H(x)$ smooth,
- normal Hamiltonian system triangular:

$$\dot{\lambda} = \vec{H}(\lambda), \quad \lambda \in T^*M \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \left(\mathrm{ad}\,\frac{dH}{dx}\right)^* x, & x \in L^*, \\ \dot{q} = q\frac{dH}{dx}, & q \in M \end{cases}$$

Normal extremals and their optimality

Exponential mapping for time t:

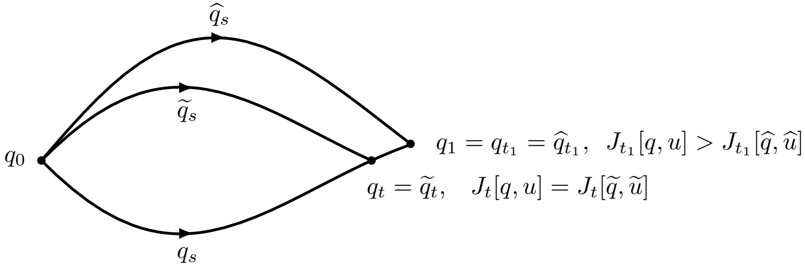
$$\mathrm{Exp}_t \; : \; N = L^* \to M, \qquad \mathrm{Exp}_t(x) = \pi \circ e^{t\vec{H}}(x, q_0) = q_t.$$

Cut time

$$t_{\mathrm{cut}}(x) = t_{\mathrm{cut}}(q_\cdot) = \sup\{t \mid q_s \text{ optimal for } s \in [0, t]\}.$$

H smooth $\quad \Rightarrow \quad t_{\mathrm{cut}}(x) > 0 \quad \forall \, x \in N.$

Maxwell points and cut points



- Maxwell point q_t: $\exists\ \widetilde{q}_s \not\equiv q_s : \quad q_t = \widetilde{q}_t,\ J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]$.
  $q_., \widetilde{q}_.$ — Maxwell pair.
- q_t is a Maxwell point $\Rightarrow$ q_s is not optimal for $s > t$.
- Time t Maxwell set in $N = L^*$:

$$\mathrm{MAX}_t = \{x \in N \mid \exists\ \widetilde{x} \in N\ :\ \mathrm{Exp}_.(x),\ \mathrm{Exp}_.(\widetilde{x}) \text{ — Maxwell pair } \}.$$

- $x \in \mathrm{MAX}_t \quad \Rightarrow \quad t_{\mathrm{cut}}(x) \leq t$

Symmetries of the exponential mapping

- Invertible $\Phi : N \to N$ is a symmetry of Exp_t if $\exists \varphi : M \to M$ s. t.

$$\Phi \circ \mathrm{Exp}_t = \mathrm{Exp}_t \circ \varphi, \qquad J_t[q_\cdot] = J_t[\varphi(q_\cdot)].$$

- Maxwell set corresponding to a group of symmetries $G = \{\Phi : N \to N\}$:

$$\mathrm{MAX}_t^G = \{x \in N \mid \exists\ \Phi \in G\ :\ \widetilde{x} = \Phi(x) \neq x, \\ \mathrm{Exp}_\cdot(x),\ \mathrm{Exp}_\cdot(\widetilde{x}) \text{ — Maxwell pair } \}.$$

- $\mathrm{MAX}_t^G \subset \mathrm{MAX}_t$.

Construction of symmetries of Exp_t

$$\dot{x} = \left(\mathrm{ad}\, \frac{d\,H}{d\,x}\right)^{*} x, \quad \dot{q} = q \frac{d\,H}{d\,x}, \qquad x \in N, \quad q \in M. \tag{1}$$

- Take a diffeomorphism $\Phi : N \to N$.
- $\forall x_0 \in N$, $\widetilde{x}_0 = \Phi(x_0) \in N$, find:
  $x_s, \widetilde{x}_s \in N$; $q_s, \widetilde{q}_s \in M$ the corresponding solutions to (1),
- Suppose that $J_t[q_\cdot] = J_t[\widetilde{q}_\cdot]$.
- Suppose that $\exists \varphi : M \to M$ such that $\widetilde{q}_t = \varphi(q_t)$.

Then Φ is a symmetry of Exp_t.

Estimates of cut time and conjugate times

- $\mathrm{Exp}_{t_1} : N = L^* \to M$ onto, with singularities and multiple points.
- G — a group of symmetries of Exp_{t_1},
  $t^G_{\mathrm{Max}}(x) = \min\{t > 0 \mid x \in \mathrm{MAX}^G_{t_1}\}$.
- $t_{\mathrm{cut}}(x) \leq t^G_{\mathrm{Max}}(x) \quad \forall x \in N$.
- $N' = \{x \in N \mid t^G_{\mathrm{Max}}(x) \leq t_1\}$,
  $\mathrm{Exp}_{t_1} : N' \to M$ onto, with singularities and multiple points.
- Suppose:

$$\text{the first conjugate time } t_{\mathrm{conj}}(x) \geq t^G_{\mathrm{Max}}(x) \quad \forall x \in N.$$

Diffeomorphic domains in the preimage and image of Exp_t

- $N' = (\cup_{i=1}^{k_N} N_i) \cup N''$, $\quad \mathrm{cl}(\cup_{i=1}^{k_N} N_i) \supset N'$,
- $\mathcal{A}_{q_0}(t_1) = (\cup_{j=1}^{k_M} M_j) \cup M''$, $\quad \mathrm{cl}(\cup_{j=1}^{k_M} M_j) \supset \mathcal{A}_{q_0}(t_1)$,
- N_i, M_j open, connected, simply connected,
  $\mathrm{Exp}_{t_1} : N_i \to M_{j(i)}$ nondegenerate, proper $\quad \Rightarrow \quad$ diffeomorphic,
- $\mathrm{Exp}_{t_1}(N'') = M''$.
- $k_N \geq k_M$.

Group G sufficient for deciding optimality

$k_N = k_M \quad \Rightarrow$

- $i \mapsto j(i)$ bijective, $\mathrm{Exp}_{t_1}(N_i) = M_{j(i)}$,
- $\forall q \in M_j \ \exists! x \in N', \ x \in N_i, \ j = j(i): \quad \mathrm{Exp}_{t_1}(x) = q$.
- $\quad \Rightarrow \quad q_s = \mathrm{Exp}_s(x)$ is optimal.

Group G not sufficient for deciding optimality

$k_N > k_M \quad \Rightarrow$

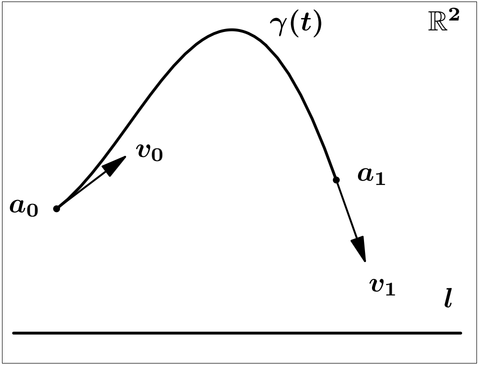
- $i \mapsto j(i)$ not bijective, $\mathrm{Exp}_{t_1}(N_i) = M_{j(i)}$,
- $\forall q \in M_j \ \exists x_{i_1} \in N_{i_1}, \dots x_{i_l} \in N_{i_l}$:
  $\mathrm{Exp}_{t_1}(x_{i_1}) = \cdots = \mathrm{Exp}_{t_1}(x_{i_l}) = q$,
- $J_{t_1}[q_{i*}] = \min\{J_{t_1}[q_{i_1}], \dots, J_{t_1}[q_{i_l}]\}$,
  $q_{i_m}(s) = \mathrm{Exp}_s(x_{i_m})$, $m = 1, \dots, l$.
- $\Rightarrow \quad q_{i*}(s)$ is optimal.

Main steps for deciding controllability

1. Construction of a group G of symmetries of Exp (continuation of symmetries of adjoint system to symmetries of Exp).
2. Description of the Maxwell set Max_t^G (bounds on Maxwell times and cut time).
3. Estimate of the first conjugate time t_{conj} w.r.t. the first Maxwell time t_{Max}^G.
4. Study of the global structure of Exp (diffeomorphic domains in the preimage and image).

Euler's elastic problem

Stationary configurations of elastic rod



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.

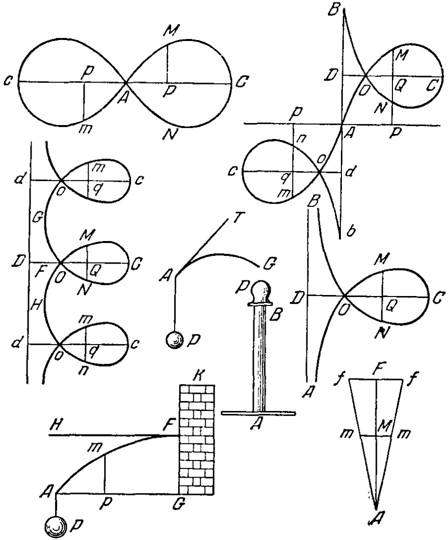
Find: $\gamma(t)$, $t \in [0, t_1]$:

$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot{\gamma}(0) = v_0$, $\dot{\gamma}(t_1) = v_1$. $|\dot{\gamma}(t)| \equiv 1 \quad \Rightarrow \quad t_1 = l$

Elastic energy $J = \frac{1}{2}\int_0^{t_1} k^2\, dt \to \min$, $k(t)$ — curvature of $\gamma(t)$.

1744: Leonhard Euler

- Problem of calculus of variations
- Euler-Lagrange equation
- Reduction to quadratures
- Qualitative analysis of the integrals
- Types of solutions (elasticae)



Optimal control problem

$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta, \qquad \dot{\theta} = u,$$

$$q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R},$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \; q(t_1) = q_1 = (x_1, y_1, \theta_1), \; t_1 \text{ fixed},$$

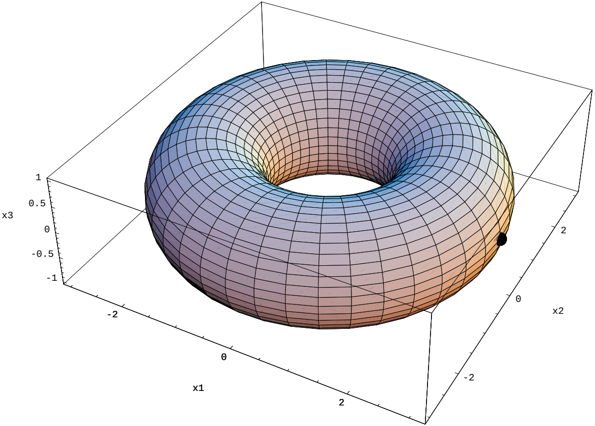
$$J = \frac{1}{2} \int_0^{t_1} u^2 \, dt \to \min .$$

Left-invariant problem on the group of motions of a plane

$$\mathrm{E}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2, \; \theta \in S^1 \right\}.$$

Attainable set and existence of optimal solutions

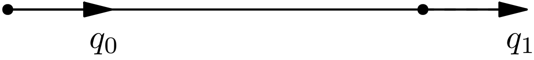
$\mathcal{A}_{\mathrm{Id}}(1) = \{(x, y, \theta) \mid x^2 + y^2 < 1 \ \forall \theta \in S^1 \text{ or } (x, y, \theta) = (1, 0, 0)\}.$



$q_1 \in \mathcal{A}_{q_0}(t_1) \quad \Rightarrow \quad \exists$ optimal $u(t) \in L_\infty$.

Abnormal extremal trajectories

$u(t) \equiv 0 \quad \Rightarrow \quad \theta \equiv 0, \qquad x = t, \qquad y \equiv 0$



$J = 0 =$ min $\quad \Rightarrow$

$\Rightarrow \quad$ abnormal extremal trajectories optimal for $t \in [0, t_1]$

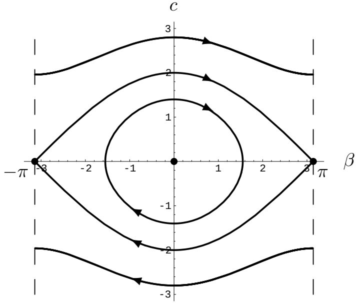
Unique trajectory from q_0 to $(t_1, 0, 0) \in \partial \mathcal{A}_{q_0}(t_1)$.

Normal Hamiltonian system of PMP

$$\dot{h}_1 = -h_2 h_3, \quad \dot{h}_2 = h_3, \quad \dot{h}_3 = h_1 h_2,$$
$$\dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \quad \dot{\theta} = h_2.$$

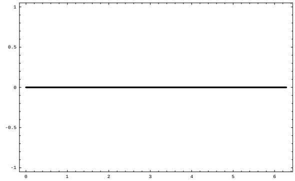
$$r^2 = h_1^2 + h_3^2 \equiv \text{const} \quad \Rightarrow \quad h_1 = -r\cos\beta, \; h_3 = -r\sin\beta$$

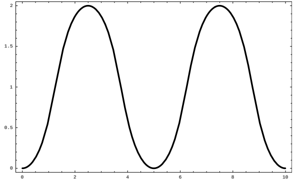
$$\begin{cases} \dot{\beta} = c, \\ \dot{c} = -r\sin\beta \end{cases}$$

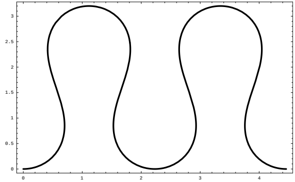


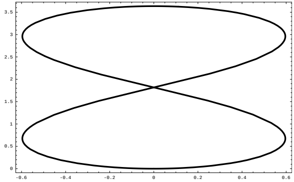
$\theta(t)$, $x(t)$, $y(t)$ parametrized by Jacobi's functions cn, sn, dn, E.

Euler elasticae

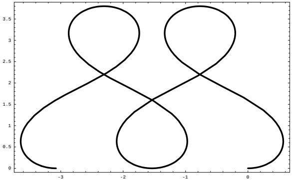


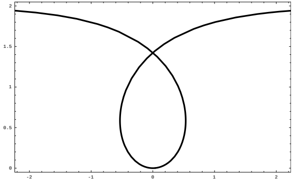


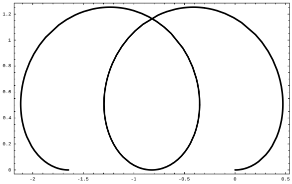


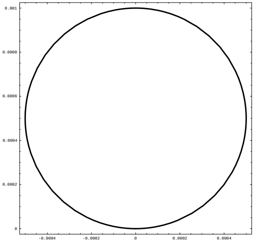


Euler elasticae

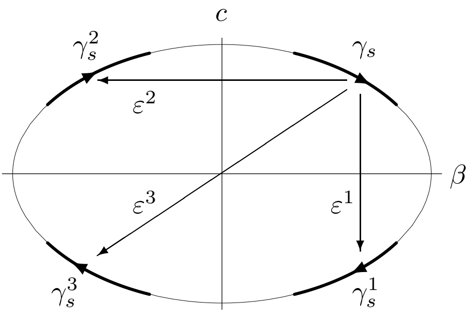








Symmetries of the pendulum $\ddot{\beta} = -r \sin \beta$

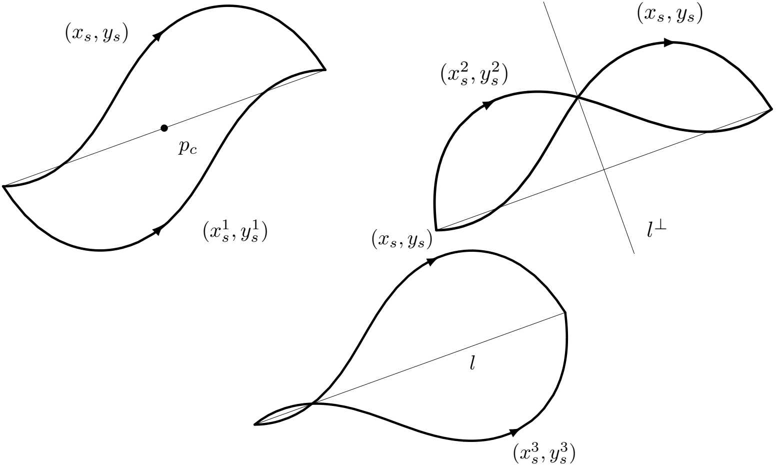


Group of symmetries $G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$

	ε^1	ε^2	ε^3
ε^1	Id	ε^3	ε^2
ε^2	ε^3	Id	ε^1
ε^3	ε^2	ε^1	Id

Group of symmetries of exponential mapping

Action of reflections ε^1, ε^2, ε^3 on elasticae:



Group $G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$ continued to $\varphi : M \to M$
and to symmetries of Exp_t.

Maxwell points corresponding to the group G

Fixed points of reflections $\varepsilon^i \in G \quad \Rightarrow \quad$ Maxwell times:

$$t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \quad n = 1, 2, \dots$$

T = period of pendulum $\quad \Rightarrow$

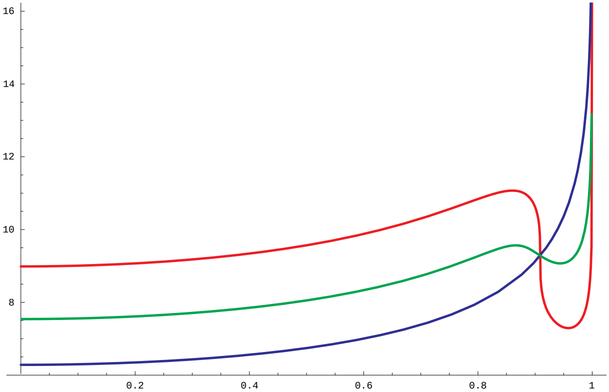
$$t^n_{\varepsilon^1} = nT, \qquad \left(n - \frac{1}{2}\right) T < t^n_{\varepsilon^2} < \left(n + \frac{1}{2}\right) T.$$

Upper bound of cut time:

$$t_{\text{cut}} \leq t^G_{\text{Max}} = \min(t^1_{\varepsilon^1}, t^1_{\varepsilon^2}) \leq T.$$

Estimate of conjugate time in elastic problem

- Non-inflectional elasticae $\Rightarrow$ no conjugate points.
- Inflectional elasticae $\Rightarrow$ $t_{\text{conj}} \in [t^1_{\varepsilon^1}, t^1_{\varepsilon^2}]$:



Plots of $t = t^1_{\varepsilon^1}$, $t = t^1_{\varepsilon^2}$, $t = t_{\text{conj}}$.

Global structure of Exp in Euler's elastic problem

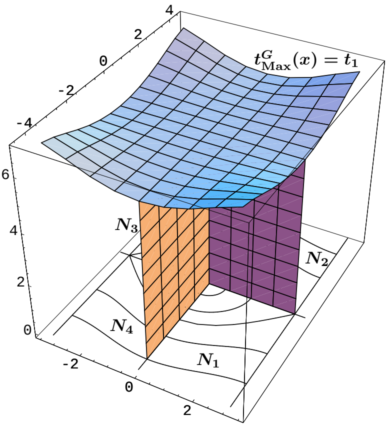


Figure: $N' = \cup_{i=1}^4 N_i \cup N''$

$\xrightarrow{\mathrm{Exp}_{t_1}}$

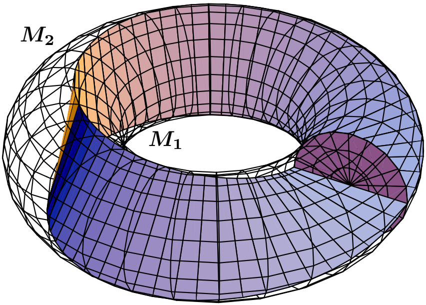
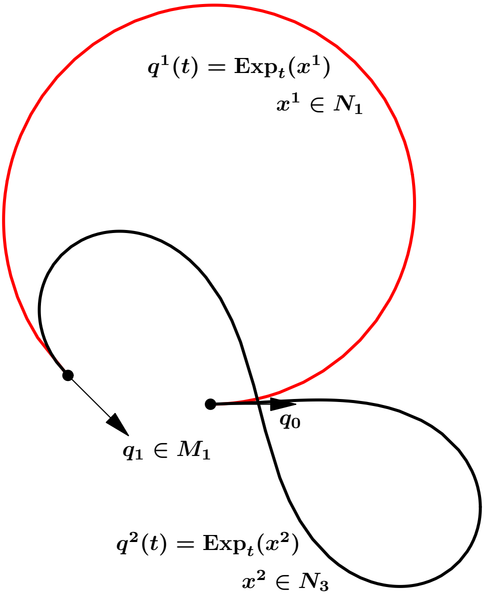


Figure: $\mathcal{A}_{q_0}(t_1) = \cup_{j=1}^2 M_j \cup M''$

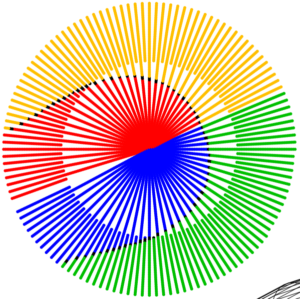
$\mathrm{Exp}_{t_1} \ : \ N_1, \ N_3 \to M_1$ diffeo, $\qquad \mathrm{Exp}_{t_1} \ : \ N_2, \ N_4 \to M_2$ diffeo

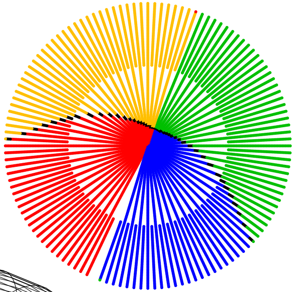
Deciding optimality

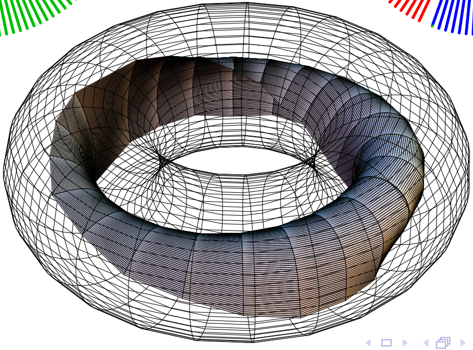


? : $J[q^1_\cdot] \lesseqgtr J[q^2_\cdot]$

Additional Maxwell stratum in Euler's elastic problem







Sub-Riemannian problem on $\mathrm{E}(2)$

$$
\begin{aligned}
&\dot{x} = u \cos\theta, \\
&\dot{y} = u \sin\theta, \\
&\dot{\theta} = v, \\
&q = (x, y, \theta) \in \mathrm{E}(2) \simeq \mathbb{R}^2_{x,y} \times S^1_\theta, \\
&(u, v) \in \mathbb{R}^2, \\
&q(0) = q_0, \qquad q(t_1) = q_1, \\
&l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .
\end{aligned}
$$

Extremals, symmetries, and cut time

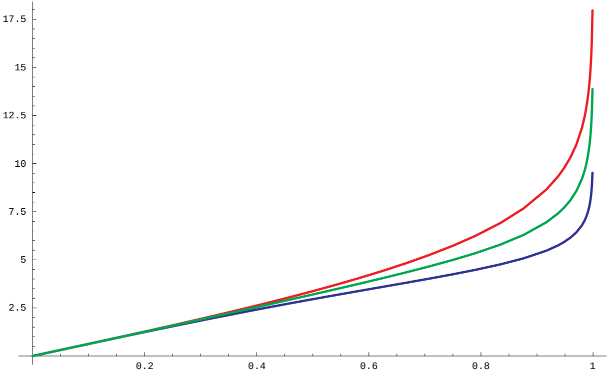
- Normal Hamiltonian system of PMP

$$\dot\gamma = c, \quad \dot c = -\sin\gamma, \qquad \gamma \in 2\,S^1 = [0, 4\pi], \quad c \in \mathbb{R},$$
$$\dot x = \sin\frac{\gamma}{2}\cos\theta, \quad \dot y = \sin\frac{\gamma}{2}\sin\theta, \quad \dot\theta = -\cos\frac{\gamma}{2}$$

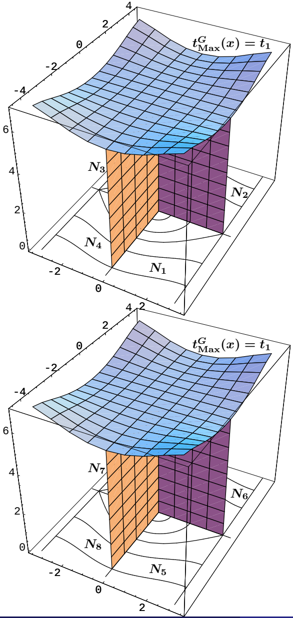
  integrated in Jacobi's functions cn, sn, dn, E.
- Symmetries of Exp:
  reflections of the pendulum $G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$.
- $t^G_{\mathrm{Max}} = \min(t^1_{\varepsilon^1}, t^1_{\varepsilon^2})$ completely described.
- t_{conj} estimated.
- G sufficient to decide controllability.
- $t_{\mathrm{cut}} = t^G_{\mathrm{Max}}$.

Estimate of conjugate time in SR problem on $\mathrm{E}(2)$

- non-inflectional trajectories $\Rightarrow$ $t^1_{\varepsilon^2} = +\infty$, $t_{\text{conj}} + \infty$.
- inflectional trajectories $\Rightarrow$ $t_{\text{conj}} \in [t^1_{\varepsilon^2}, t^1_{\varepsilon^1}]$:



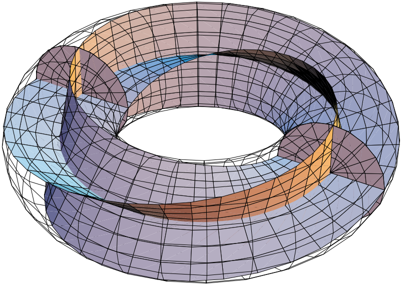
Plots of $t = t^1_{\varepsilon^1}$, $t = t^1_{\varepsilon^2}$, $t = t_{\text{conj}}$.



$\xrightarrow{\text{Exp}}$

Exp : $N_i \to M_i$ diffeo,

$i = 1, \ldots, 8.$



Nilpotent $(2, 3, 5)$ sub-Riemannian problem

- Lie algebra $L = \mathrm{span}(X_1, \dots, X_5)$ with

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5,$$

- M — the corresponding connected simply connected Lie group,
- SR structure on M: $\Delta = \mathrm{span}(X_1, X_2)$, $\langle X_i, X_j \rangle = \delta_{ij}$, $i, \ j = 1, \ 2$,
- SR problem:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in M, \quad u = (u_1, u_2) \in U = \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Optimal control problem

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q = (x, y, z, v, w) \in M = \mathbb{R}^5, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0 = 0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2 + y^2}{2}\frac{\partial}{\partial w},$$

$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2 + y^2}{2}\frac{\partial}{\partial v}.$$

Normal extremals

Normal Hamiltonian system of PMP:

$$\dot{\lambda} = \vec{H}(\lambda) \quad \Leftrightarrow \quad \begin{cases} \dot{\theta} = c, \\ \dot{c} = -\alpha \sin(\theta - \beta), \\ \dot{\alpha} = 0, \\ \dot{\beta} = 0, \\ \dot{q} = (\cos\theta)\, X_1(q) + (\sin\theta)\, X_2(q), \end{cases}$$

$$\lambda = (\theta, c, \alpha, \beta; q) \in T^* M \cap \{H(\lambda) = 1/2\}.$$

(x_t, y_t) Euler elasticae

Symmetries of the exponential mapping:

- rotations $e^{r\vec{h}_0}$: $\beta \mapsto \beta + r$,
- reflections of the pendulum ε^i, $i = 1, 2, 3$.

Maxwell time and cut time

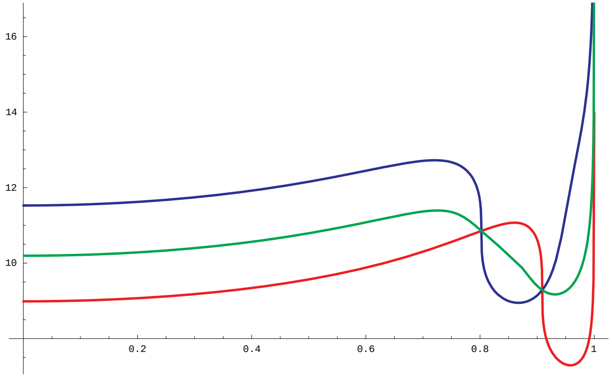
- $G = \langle e^{r\vec{h}_0}, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$ group of symmetries of Exp,
- MAX^G computed,
- for any extremal trajectory $q_\cdot$, the first Maxwell time is evaluated :

$$t^G_{\mathrm{Max}}(q_\cdot) \in (0, +\infty].$$

- Conjecture: $t_{\mathrm{cut}}(q_\cdot) = t^G_{\mathrm{Max}}(q_\cdot)$.

Estimate of conjugate time in (2,3,5) SR problem

- non-inflectional elasticae $\Rightarrow$ $t^1_{\varepsilon^1} = +\infty$, $t_{\text{conj}} \geq t^1_{\varepsilon^2}$.
- inflectional elasticae $\Rightarrow$ $t_{\text{conj}} \in [t^1_{\varepsilon^1}, t^1_{\varepsilon^2}]$:



Plots of $t = t^1_{\varepsilon^1}$, $t = t^1_{\varepsilon^2}$, $t = t_{\text{conj}}$.

The ball-plate problem

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{R} = R(vA_1 - uA_2),$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$q = (x, y, R) \in \mathbb{R}^2 \times \mathrm{SO}(3), \qquad (u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0 = \mathrm{Id}, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min.$$

Results of V.Jurdjevic (1993)

- vertical subsystem of the Hamiltonian system of PMP:

$$\dot{\theta} = c, \qquad \dot{c} = -A \sin \theta, \qquad \dot{\alpha} = 0,$$
$$\dot{x} = \cos(\theta + \alpha), \qquad \dot{y} = \sin(\theta + \alpha),$$
$$\dot{R} = R(\sin(\theta + \alpha)A_1 - \cos(\theta + \alpha)A_2).$$

- projections (x, y) of extremal trajectories are Euler elasticae,
- Hamiltonian system of PMP integrable in quadratures (elliptic functions and their integrals).

Symmetries and Maxwell points in the ball-plate problem

- Symmetries of Exp:

$$G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3, e^{r\vec{h}_0} \rangle, \qquad e^{r\vec{h}_0} \; : \; \theta \mapsto \theta + r.$$

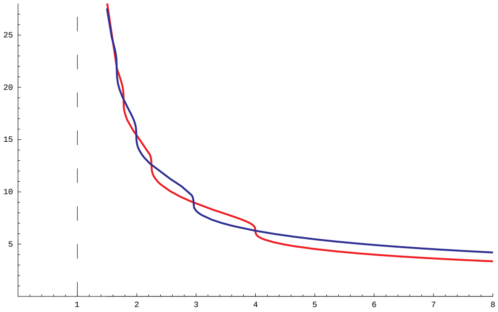
- Maxwell strata corresponding to reflections

$$\mathrm{Max}_{\varepsilon^1} = \{f_{\varepsilon^1} = 0\}, \qquad \mathrm{Max}_{\varepsilon^2} = \{f_{\varepsilon^2} = 0\}.$$

- Implicit description $t^G_{\mathrm{Max}} = \min\{t > 0 \mid f_{\varepsilon^1} f_{\varepsilon^2} = 0\}$.
- Upper bound $t_{\mathrm{cut}} \leq t^G_{\mathrm{Max}}$.
- Conjecture: $t_{\mathrm{cut}} = t^G_{\mathrm{Max}}$.

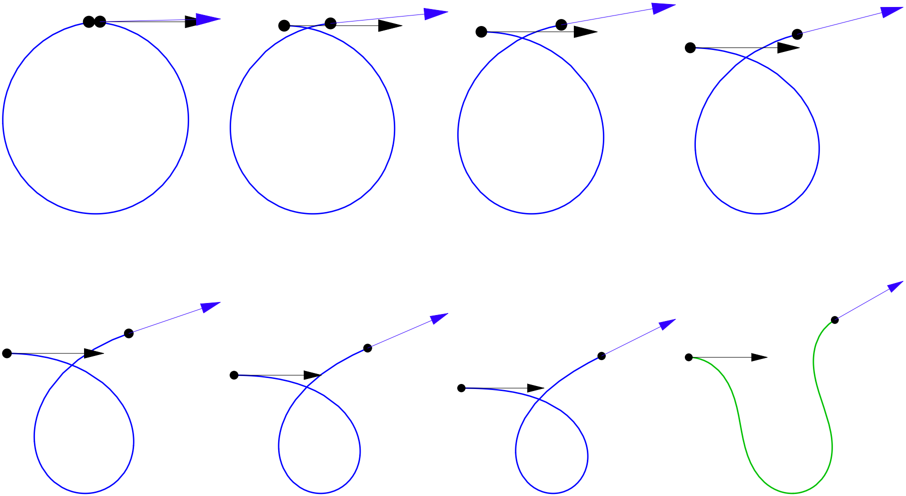
Asymptotics near the stable equilibrium of pendulum

- Sphere rolling along sinusoids of small amplitude.
- Asymptotics of Exp, f_{ε^i} computed.
- Complete description of $t^1_{\varepsilon^i}$, t^G_{Max}.
- $t_{\mathrm{conj}} \in [t^1_{\varepsilon^1}, t^1_{\varepsilon^2}]$.



Plots of $t = t^1_{\varepsilon^1}, t = t^1_{\varepsilon^2}$.

Movies with optimal curves



Publications

http://www.botik.ru/PSI/CPRC/sachkov/

[1] Yu. L. Sachkov, Exponential mapping in generalized Dido's problem, *Sbornik: Mathematics*, **194** (2003).

[2] Yu. L. Sachkov, Discrete symmetries in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 2: 235–257.

[3] Yu. L. Sachkov, The Maxwell set in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 4: 595–621.

[4] Yu. L. Sachkov, Complete description of the Maxwell strata in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 6: 901–950.

[5] Yu. L. Sachkov, Maxwell strata in Euler's elastic problem, *Journal of Dynamical and Control Systems* (accepted), available at `arXiv:0705.0614 [math.OC]`, 3 May 2007.

[6] Yu. L. Sachkov, Conjugate points in Euler's elastic problem, *Journal of Dynamical and Control Systems* (accepted), available at `arXiv:0705.1003 [math.OC]`, 7 May 2007.

[7] Yu. L. Sachkov, Optimality of Euler elasticae, *Doklady Mathematics*, **417** (2007), No. 1, pp. 23–25.