Low-dimensional left-invariant sub-Riemannian problems

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in (planned) collaboration with
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«Geometric control and sub-Riemannian geometry»

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Part 1:
Nilpotent sub-Riemannian problem on the Engel group
Problem Statement

$$\dot q = \begin{pmatrix} \dot x \\ \dot y \\ \dot z \\ \dot v \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix},$$

$$q = (x, y, z, v) \in \mathbb{R}^4, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$q(0) = q_0 = (0, 0, 0, 0)^T, \quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1)^T,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min \iff \int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min .$$

Results obtained

- Parameterisation of extremal trajectories,
- discrete symmetries and their fixed points (Maxwell points),
- bounds on conjugate time,
- diffeomeorphic domains in preimage and image of exponential mapping,
- global structure of exponential mapping,
- cut time and cut locus,
- explicit solutions for some special boundary conditions,
- reduction of optimal control problem to solving systems of algerbaic equations in Jacobi's functions,
- software for computation of sub-Riemannian length minimisers for arbitrary boundary conditions.

Known results for invariant sub-Riemannian problems on Lie groups

- Heisenberg group (A.Vershik, V.Gershkovich 1986) and its genertlizations (D.Barilari, U.Boscain; O. Myasnichenko),
- SL(2), SO(3), S^3 (U. Boscain, F. Rossi 2008),
- SE(2) (Yu.S. 2010)

- 5-D nilpotent Lie group with the growth vector $(2, 3, 5)$ (Yu.S. 2006).

Nilpotent sub-Riemannian problem on the Engel group

$$\dot q \in \mathrm{span}(X_1, X_2), \quad q(0) = q_0, \quad q(t_1) = q_1, \quad \int_0^{t_1} \langle \dot q, \dot q \rangle^{1/2}\, dt \to \min$$

$$X_1 = (1, 0, -\frac{y}{2}, 0)^T, \quad X_2 = (0, 1, \frac{x}{2}, \frac{x^2 + y^2}{2})^T, \quad \langle X_i, X_j \rangle = \delta_{ij}.$$

$$\begin{aligned}
&\mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, X_2, X_3, X_4),\\
&\dim \mathrm{Lie}(X_1, X_2)(q) = 4,\\
&[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,\\
&[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.
\end{aligned}$$

Growth vector (2, 3, 4).

Nilpotent approximation to nonholonomic systems in 4-D space with 2-D control (e.g. a mobile robot with trailer).

Existence of optimal controls and PMP

- $X_1(q), \dots, X_4(q)$ linearly independent $\forall q \in \mathbb{R}^4 \Rightarrow$ complete controllability by Rashevskii-Chow theorem.
- Existence of optimal trajectories (sub-Riemannian length minimizers) follows from Filippov's theorem.
- Pontryagin maximum principle
- Abnormal extremal trajectories are normal ($\Rightarrow$ smooth).

Normal Hamiltonian system of Pontryagin maximum principle

$$
\begin{aligned}
\dot\theta &= c, & \theta &\in S^1,\\
\dot c &= -\alpha \sin\theta, & c &\in \mathbb{R},\\
\dot\alpha &= 0, & \alpha &\in \mathbb{R},\\
\dot q &= \cos\theta \; X_1(q) + \sin\theta \; X_2(q).
\end{aligned}
$$

$$
E = \frac{c^2}{2} - \alpha\cos\theta \in [-|\alpha|, +\infty).
$$

Equation of pendulum and physical meaning of parameter α

$$\ddot{\theta} = -\alpha \sin\theta, \qquad \alpha = \frac{g}{L} = \text{const} \in \mathbb{R}$$

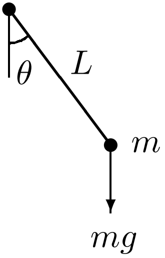


Figure: Pendulum for $\alpha > 0$

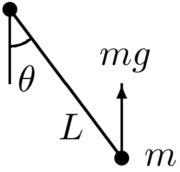


Figure: Pendulum for $\alpha < 0$

Stratification of phase cylinder of pendulum

$$C = T^*_{q_0}M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\}.$$
$$C = \cup_{i=1}^{7} C_i, \quad C_i \cap C_j = \emptyset,\ i \neq j.$$

$$C_i^+ = C_i \cap \{\alpha > 0\}, \qquad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\},$$
$$C_{i+}^{\pm} = C_i^{\pm} \cap \{c > 0\}, \qquad C_{i-}^{\pm} = C_i^{\pm} \cap \{c < 0\}, \qquad i \in \{2, 3\}.$$

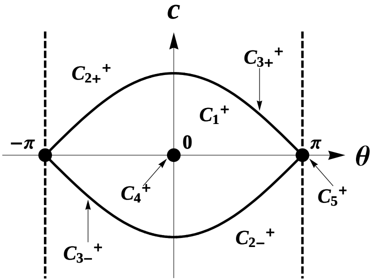


Figure: Stratification for $\alpha > 0$

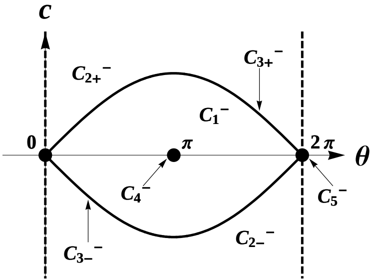
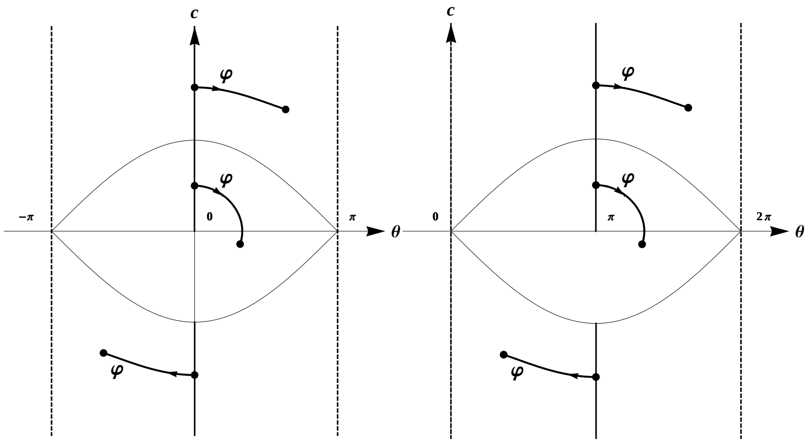


Figure: Stratification for $\alpha < 0$

Elliptic coordinates (φ, k) in phase cylinder of pendulum



$$\dot{\varphi} = 1, \quad \dot{k} = 0, \quad \dot{\alpha} = 0.$$

Parameterisation of extremal trajectories for $\alpha = 1$

$\lambda \in C_1^+$ (oscillations of pendulum) $\Rightarrow$

$$\begin{aligned}
x_t &= 2k(\operatorname{cn}\varphi_t - \operatorname{cn}\varphi), \\
y_t &= 2(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) - t, \\
z_t &= 2k(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi - \frac{y_t}{2}(\operatorname{cn}\varphi_t + \operatorname{cn}\varphi)), \\
v_t &= \frac{y_t^3}{6} + 2k^2 \operatorname{cn}^2 \varphi y_t - 4k^2 \operatorname{cn}\varphi(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi) + \\
&\quad + 2k^2 \left(\frac{2}{3} \operatorname{cn}\varphi_t \operatorname{dn}\varphi_t \operatorname{sn}\varphi_t - \frac{2}{3} \operatorname{cn}\varphi \operatorname{dn}\varphi \operatorname{sn}\varphi + \frac{1-k^2}{3k^2} t + \right. \\
&\quad \left. \frac{2k^2-1}{3k^2}(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) \right).
\end{aligned}$$

Symmetries of Hamiltonian system

Dilation of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta, \frac{c}{\sqrt{\alpha}}, 1, \sqrt{\alpha}x, \sqrt{\alpha}y, \alpha z, \alpha^{\frac{3}{2}}v, \sqrt{\alpha}t),$$

$$(\varphi, k, \alpha) \mapsto (\sqrt{\alpha}\varphi, k, 1).$$

Inversion of α:

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t),$$

$$(\varphi, k, \alpha) \mapsto (\varphi, k, -\alpha).$$

Parameterisation of extremal trajectories in generic case for $\lambda \in \cup_{i=1}^{3} C_i$

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = (\frac{s_1}{\sigma} x_{\sigma t}, \frac{s_1}{\sigma} y_{\sigma t}, \frac{1}{\sigma^2} z_{\sigma t}, \frac{s_1}{\sigma^3} v_{\sigma t})(\sigma\varphi, k, 1),$$

where $\sigma = \sqrt{|\alpha|}$, $s_1 = \operatorname{sgn} \alpha$.

General case for $\alpha \neq 0$

$$
\begin{aligned}
&\lambda \in C_1 \Rightarrow \\
x_t &= \frac{2k\sigma}{\alpha}(\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)),\\
y_t &= \frac{2\sigma}{\alpha}(\operatorname{E}(\sigma\varphi_t) - \operatorname{E}(\sigma\varphi)) - \operatorname{sgn}\alpha t,\\
z_t &= \frac{2k}{|\alpha|}(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi) -\\
&\quad - \frac{\sigma k y_t}{2\alpha}(\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi))),\\
v_t &= \ldots
\end{aligned}
$$

Parameterisation of extremal trajectories for degenerate cases

$$\lambda \in C_4 \ \Rightarrow \ x_t = 0, \quad y_t = t \ \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = \frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_5 \ \Rightarrow \ x_t = 0, \quad y_t = -t \ \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = -\frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_6 \ \Rightarrow$$

$$x_t = \frac{\cos(ct+\theta) - \cos\theta}{c}, \quad y_t = \frac{\sin(ct+\theta) - \sin\theta}{c},$$

$$z_t = \frac{ct - \sin(ct)}{2c^2}, \quad v_t = -\frac{2c\cos\theta\, t - 4\sin(ct+\theta) + \sin(2ct+\theta)}{4c^3}.$$

$$\lambda \in C_7 \ \Rightarrow \ x_t = -t \sin\theta, \quad y_t = t \cos\theta, \quad z_t = 0, \quad v_t = \frac{\cos\theta}{6} t^3.$$

Exponential mapping, Maxwell points, and cut points

$$
\begin{aligned}
&\mathrm{Exp} : C \times \mathbb{R}_+ \to M = \mathbb{R}^4, \\
&\mathrm{Exp}(\lambda, t) = q_t = (x_t, y_t, z_t, v_t), \\
&\lambda = (\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+, \quad q_t \in M.
\end{aligned}
$$

$$
\mathrm{MAX} = \{(\lambda, t) \mid \exists \tilde{\lambda} \neq \lambda, \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(\tilde{\lambda}, t)\},
$$

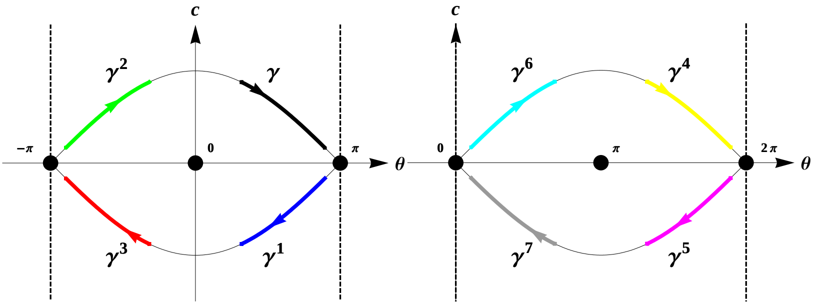
$$
\begin{aligned}
&t_{cut}(\lambda) = \sup\{t > 0 \mid \mathrm{Exp}(\lambda, s) \text{ optimal for } s \in [0, t]\}, \\
&t_{cut}(\lambda) \leq t \text{ for any } (\lambda, t) \in \mathrm{MAX}.
\end{aligned}
$$

Symmetry group of exponential mapping

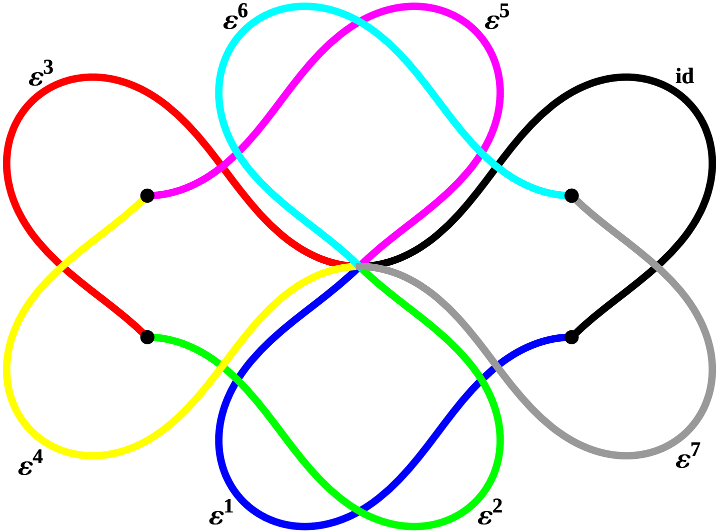
G	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
ε^1	Id	ε^3	ε^2	ε^5	ε^4	ε^7	ε^6
ε^2		Id	ε^1	ε^6	ε^7	ε^4	ε^5
ε^3			Id	ε^7	ε^6	ε^5	ε^4
ε^4				Id	ε^1	ε^2	ε^3
ε^5					Id	ε^3	ε^2
ε^6						Id	ε^1
ε^7							Id

Table: Multiplication in $G = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6, \varepsilon^7\}$

Reflection of trajectories of pendulum



Reflection of Euler elasticae



Reflections as symmetires of Exp

Proposition

All mappings ε^i are symmetries of exponential mapping, $i = 1, \dots, 7$, i. e.

$$\varepsilon^i \circ \mathrm{Exp}(\theta, c, \alpha, t) = \mathrm{Exp} \circ \varepsilon^i(\theta, c, \alpha, t),$$
$$(\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+.$$

$$\mathrm{MAX}^i = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t)\},$$
$$\lambda = (\theta, c, \alpha), \quad \lambda^i = (\theta^i, c^i, \alpha^i) = \varepsilon^i(\lambda).$$

Fixed points of reflections ε^i in image of exponential mapping

$$\mathrm{Exp}(\lambda^i, t) = \mathrm{Exp}(\lambda, t) \iff \varepsilon^i(q_t) = q_t.$$

Lemma

1. $\varepsilon^1(q) = q \iff z = 0,$
2. $\varepsilon^2(q) = q \iff x = 0,$
3. $\varepsilon^3(q) = q \iff x^2 + z^2 = 0,$
4. $\varepsilon^4(q) = q \iff x^2 + y^2 + v^2 = 0,$
5. $\varepsilon^5(q) = q \iff x^2 + y^2 + z^2 + v^2 = 0,$
6. $\varepsilon^6(q) = q \iff y^2 + (2v - xz)^2 = 0,$
7. $\varepsilon^7(q) = q \iff y^2 + z^2 + v^2 = 0.$

Fixed points of reflections ε^i in preimage of exponential mapping

Proposition

If $(\lambda, t) \in C \times \mathbb{R}_+$, $\varepsilon^i(\lambda, t) = (\lambda^i, t)$ *then:*

1. $\lambda^1 = \lambda \iff \begin{cases} \operatorname{cn} \tau = 0 \text{ if } \lambda \in C_1 \\ \text{is impossible if } \lambda \in C_2 \cup C_3 \cup C_6 \end{cases}$
2. $\lambda^2 = \lambda \iff \begin{cases} \operatorname{sn} \tau = 0 \text{ if } \lambda \in C_1 \\ \operatorname{sn} \tau \operatorname{cn} \tau = 0 \text{ if } \lambda \in C_2 \\ \tau = 0 \text{ if } \lambda \in C_3 \\ 2\theta + ct = 2\pi n \text{ if } \lambda \in C_6 \end{cases}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi_t}{2k}.$$

Complete description of Maxwell sets for $\varepsilon^1, \varepsilon^2$

Theorem

1. $\mathrm{MAX}^1 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = p_z^n(k), n \in \mathbb{N}, \mathrm{cn}(\tau) \neq 0\}$,
2. $\mathrm{MAX}^1 \cap N_2 = \mathrm{MAX}^1 \cap N_3 = \mathrm{MAX}^1 \cap N_6 = \emptyset$,
3. $\mathrm{MAX}^2 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = 2Kn, n \in \mathbb{N}, \mathrm{sn}(\tau) \neq 0\}$,
4. $\mathrm{MAX}^2 \cap N_2 = \{(\lambda, t) \in N_2 \mid p = Kn, n \in \mathbb{N}, \mathrm{sn}(\tau)\,\mathrm{cn}(\tau) \neq 0\}$,
5. $\mathrm{MAX}^2 \cap N_3 = \emptyset$,
6. $\mathrm{MAX}^2 \cap N_6 = \{(\lambda, t) \in N_6 \mid tc = 2\pi n, \theta \neq \pi k, n, k \in \mathbb{Z}\}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \qquad \Rightarrow \quad p = \frac{\sigma t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \qquad \Rightarrow \quad p = \frac{\sigma t}{2k}.$$

$p_z^n(k) > 0$ — *n–th root of* $\mathrm{dn}(p)\,\mathrm{sn}(p) + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}(p) = 0$.

First Maxwell time and cut time

$$\lambda \in C_1 \Rightarrow t^1_{\mathrm{MAX}} = \min(2p^1_z, 4K)\sigma,$$
$$\lambda \in C_2 \Rightarrow t^1_{\mathrm{MAX}} = 2Kk\sigma,$$
$$\lambda \in C_6 \Rightarrow t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|},$$
$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \Rightarrow t^1_{\mathrm{MAX}} = +\infty.$$

Theorem

For any $\lambda \in C$

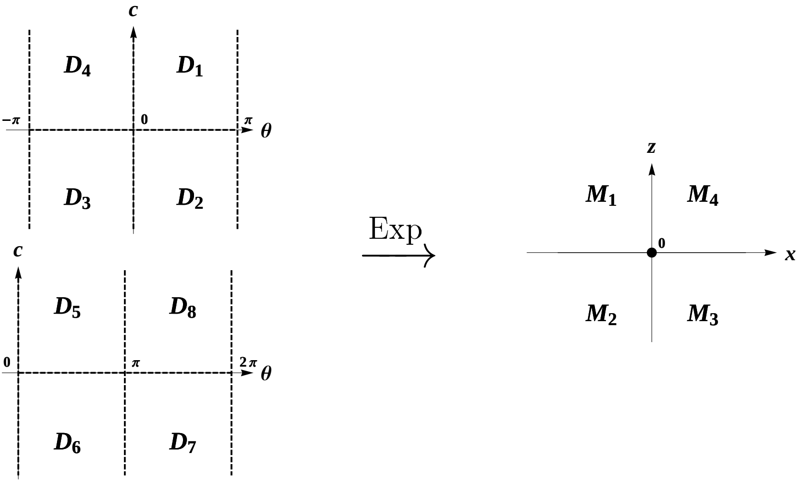
$$t_{cut}(\lambda) = t^1_{\mathrm{MAX}}(\lambda).$$

Decomposition in preimage of exponential mapping

$$
\begin{aligned}
C &= \cup_{i=1}^{8} D_i, \\
D_1 \cap C_1 &= \{\tau \in (0, K), p \in (0, p^1_{min}), k \in (0, 1)\}, \\
D_1 \cap C_2 &= \{\tau \in (0, K), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\}, \\
D_2 \cap C_1 &= \{\tau \in (K, 2K), p \in (0, p^1_{min}), k \in (0, 1)\}, \\
D_2 \cap C_2 &= \{\tau \in (-K, 0), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\}, \\
D_3 \cap C_1 &= \{\tau \in (2K, 3K), p \in (0, p^1_{min}), k \in (0, 1)\}, \\
D_3 \cap C_2 &= \{\tau \in (0, K), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\}, \\
D_4 \cap C_1 &= \{\tau \in (3K, 4K), p \in (0, p^1_{min}), k \in (0, 1)\}, \\
D_4 \cap C_2 &= \{\tau \in (-K, 0), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\},
\end{aligned}
$$

where $p^1_{min} = \min(p^1_z, 2K)$.

Correspondence between domains in preimage and image of exponential mapping



$\mathrm{Exp} : D_i \to M_i$ and $\mathrm{Exp} : D_{i+4} \to M_i$ are diffeomorphisms for $i \in \{1 \ldots 4\}$.

Conjugate points

$$d_\nu \operatorname{Exp} \, : \, T_\nu N \to T_{q_t} M \text{ is degenerate,}$$

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)}(\nu) = 0.$$

$$t^1_{\mathrm{conj}} = \min \{ t > 0 \mid t \text{ conjugate time along } \operatorname{Exp}(\lambda, s), \; s \geq 0 \}.$$

Theorem

For any $\lambda \in C$

$$t^1_{\mathrm{MAX}}(\lambda) \leq t^1_{\mathrm{conj}}(\lambda).$$

Numerical solution: reduction to system of equations

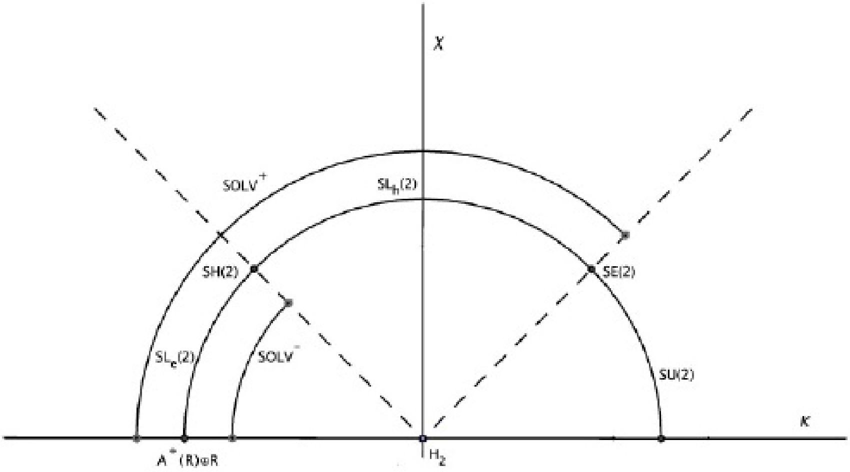
$Y = \frac{y_t}{x_t}, Z = \frac{z_t}{x_t^2}, V = \frac{v_t}{x_t^3}$ do not depend on α.

$$Y_1 = \frac{y_1}{x_1}, \quad z_1 = \frac{z_1}{x_1^2}, \quad V_1 = \frac{v_1}{x_1^3}.$$

$$\left\{ \begin{array}{l} Y(\tau, p, k) = Y_1, \\ Z(\tau, p, k) = Z_1, \\ V(\tau, p, k) = V_1. \end{array} \right.$$

Software for solving the system $\Rightarrow$ computation of sub-Riemannian length minimizers

What next?
Invariant sub-Riemannian structures on 3D Lie groups (A.Agrachev)



Part 2:
Image inpainting, neurogeometry of vision, and sub-Riemannian geometry

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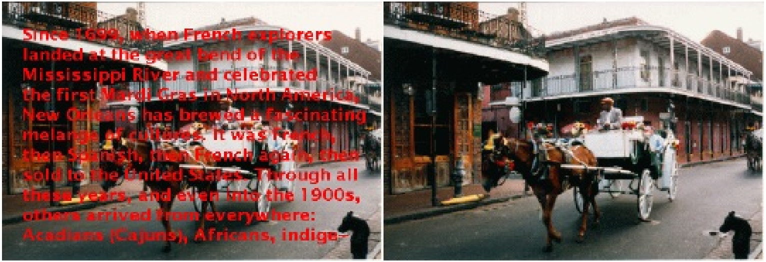
Plan of part 2

- Image inpainting
- Model of the primary visual cortex of a human brain (J.Petitot, G.Citti, A.Sarti).
- Problems of sub-Riemannian geometry (A.Agrachev, U.Boscain, F.Rossi) and their solution (Yu.S.)
- Image inpainting via sub-Riemannian length minimizers (Yu.S., A.Ardentov, A.Mashtakov)
- Curve cuspless reconstruction (U.Boscain, R.Duits, F.Rossi, Yu.S.)
- Image inpainting via hypoelliptic diffusion (J.-P.Guthier, U.Boscain, F.Rossi).

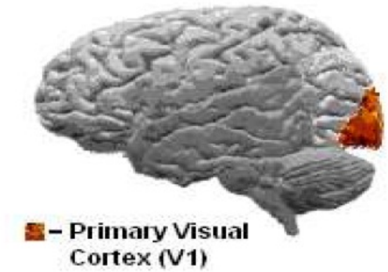
Image inpainting

M. Bertalmio, G. Sapiro, V. Caselles, C. Ballester

Image inpainting



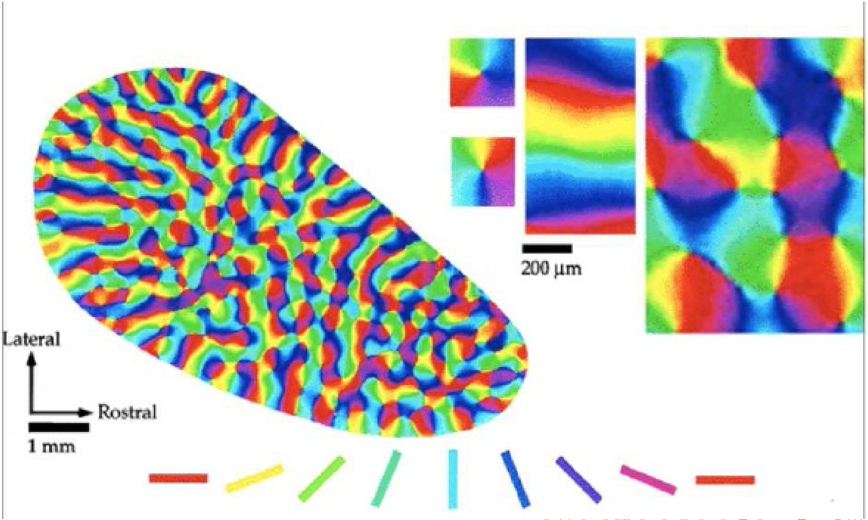
Neurophysiology of vision



A Groups of neurons of the primary visual cortex V1 of human brain are sensible both to position and orientation. Thus V1 lifts images from the plane of image $\mathbb{R}^2$ to the projective tangent bundle $PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1$.

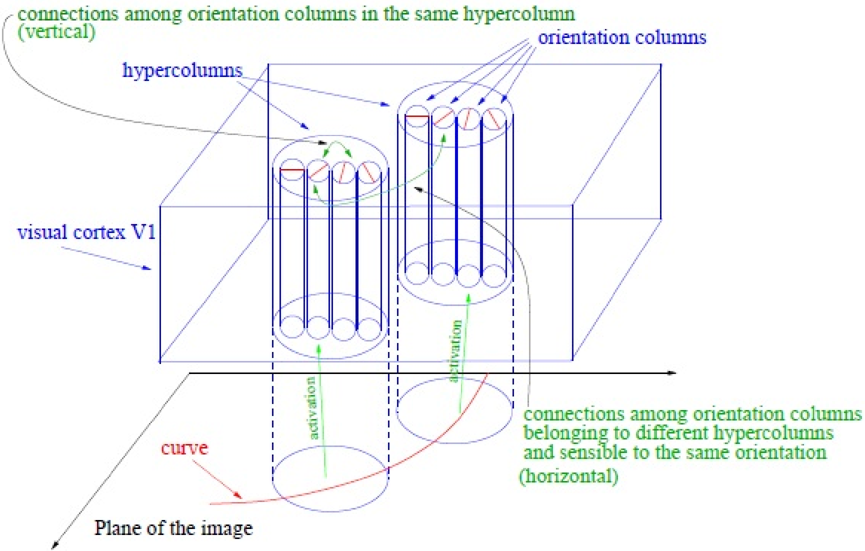
B During inpainting of images, there is minimized the activation energy for neurons not activated by the image at $PT\mathbb{R}^2$.

A1. Hubel and Wiesel (Nobel prise 1981): Groups of neurons sensible to direction



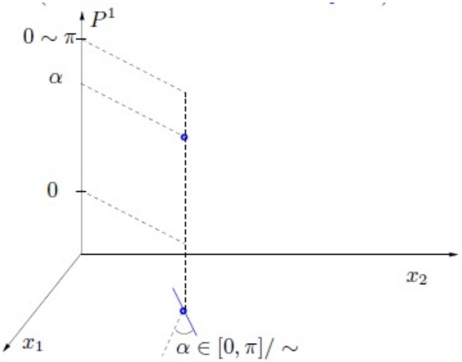
Model of visual cortex V1

"Pinwheel" model:



A2. Lift to $PT\mathbb{R}^2$

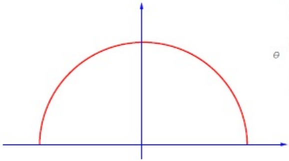
- Brain saves image as a set of positions and directions, i.e., it lifts images to $PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1$.

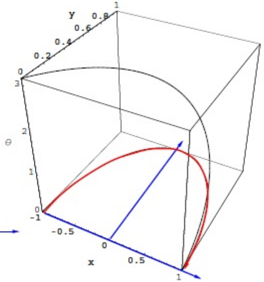


- $PT\mathbb{R}^2$ is a bundle with base $\mathbb{R}^2$ and fiber P^1.

A3. Lift of a curve

- $\mathbb{R}^2 \ni (x(t), y(t)) \mapsto (x(t), y(t), \theta(t)) \in PT\mathbb{R}^2$,
  $\theta(t) = \arctan(\dot{y}(t)/\dot{x}(t)) \in P^1 = [0, \pi]/\sim$.
  Example: $(\cos t, \sin t)$:





- any regular curve in $\mathbb{R}^2$ has a lift to $PT\mathbb{R}^2$,
- not any curve in $PT\mathbb{R}^2$ is a lift of some curve in $\mathbb{R}^2$.

A4. Which curves in $PT\mathbb{R}^2$ are lifts of planar curves?

$$\theta(t) = \arctan(\dot{y}(t)/\dot{x}(t)) \iff$$

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} =: u_2,$$

$$q = (x, y, \theta) \in PT\mathbb{R}^2, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q),$$

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

B1. Which functional is minimized?

- brain minimizes a functional (internal or external for the brain),
- when moving a hand, the brain minimizes a compromise between energy and strength of muscles (exterior functional),
- when reconstructing a curve, the brain minimizes the activation energy of neurons (interior functional),
- easily activated are the neurons close one to another both in position and in orientation (i.e., close in $PT\mathbb{R}^2$).

Problem of sub-Riemannian geometry on $PT\mathbb{R}^2$

$$\int (u_1^2 + \alpha^2 u_2^2)\, dt \to \min \iff \int \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min$$

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \quad q = (x, y, \theta) \in PT\mathbb{R}^2, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$\int_0^{t_1} \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min .$$

$$\theta \in P^1 = \mathbb{R}/(\pi\mathbb{Z}) = [0, \pi]/\sim .$$

Problem of sub-Riemannian geometry on SE(2)

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z}),\ x, y \in \mathbb{R} \right\} \cong$$

$$\cong \mathbb{R}^2 \times S^1.$$

$$\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z}) = [0, 2\pi]/\sim .$$

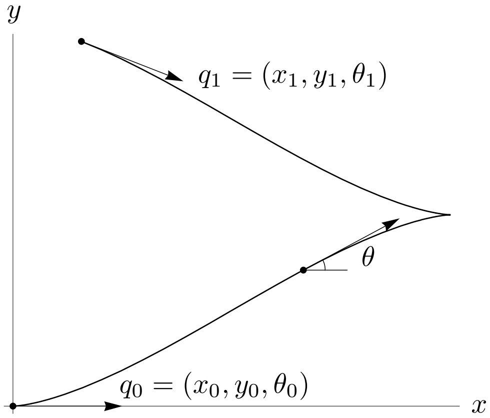
$$\dot q = u_1 X_1(q) + u_2 X_2(q), \quad q = (x, y, \theta) \in \mathrm{SE}(2), \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\int_0^{t_1} \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min .$$

Sub-Riemannian problem on the group of motions of a plane, *or* Problem on optimal motion of a mobile robot in the plane

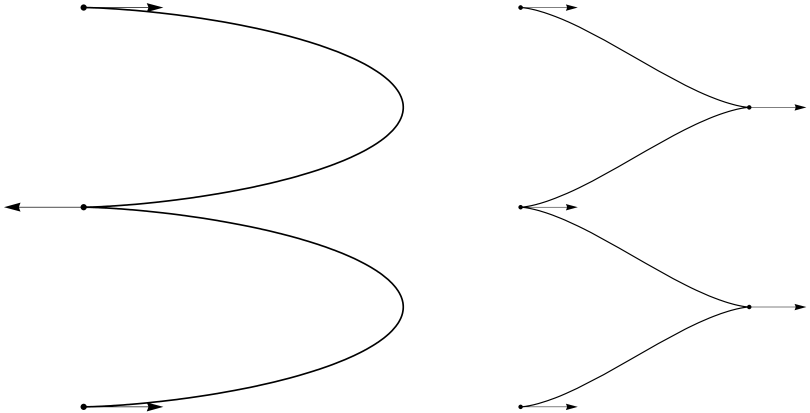


$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \alpha^2\,\dot{\theta}^2}\,dt \to \min$$

Results on sub-Riemannian problems on SE(2) and $PT\mathbb{R}^2$

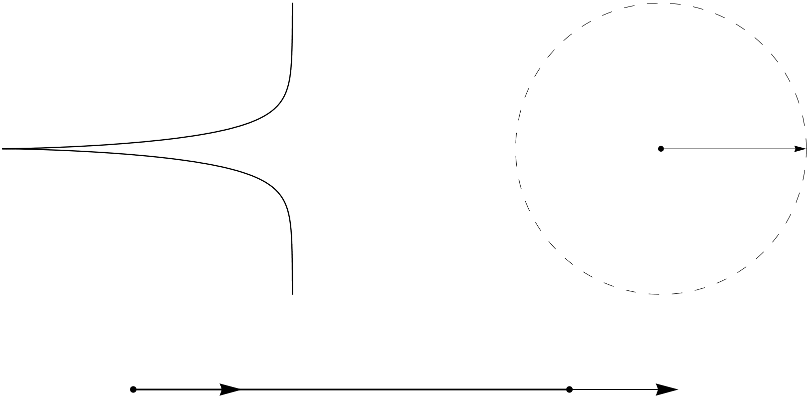
- Existence of optimal trajectories,
- Parameterisation of extremal trajectories (via Pontryagin maximum principle),
- Description of optimal trajectories:
  - Arbitrary boundary conditions $\Rightarrow$ reduction to systems of algebraic equations,
  - Special boundary conditions $\Rightarrow$ explicit solutions,
- Structure of optimal synthesis and Maxwell set,
- Sub-Riemannian spheres,
- Applications: reconstruction of curves, Parallel software for image inpainting.

Sub-Riemannian problem on SE(2): generic extremal trajectories



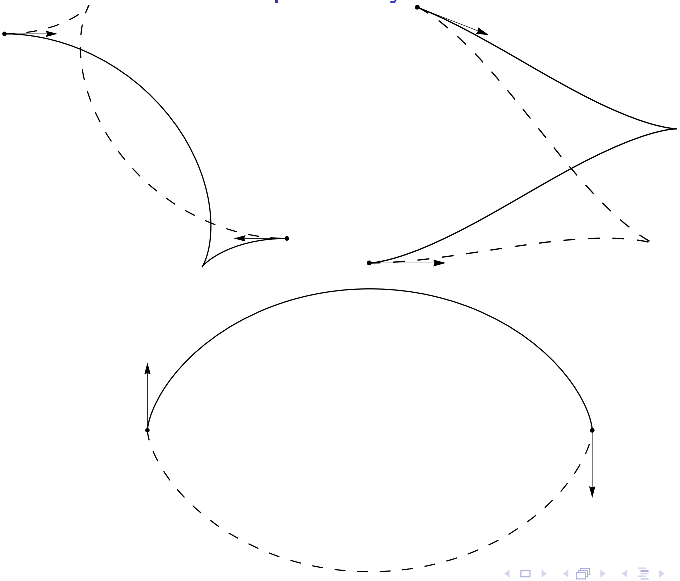
Parameterisation by Jacobi's functions cn, sn, dn, E.

Sub-Riemannian problem on SE(2): special extremal trajectories



Parameterisation by elementary functions.

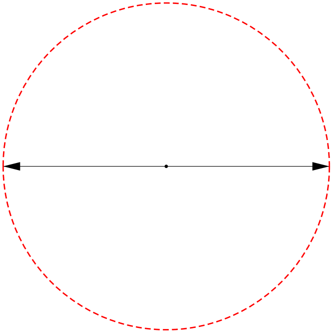
Maxwell points on extremal trajectories: optimality loss



Optimal trajectories

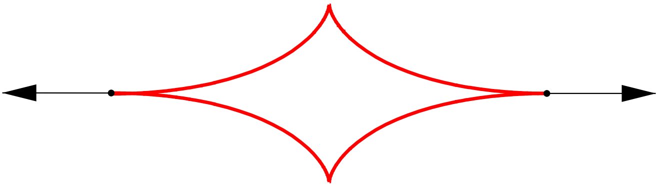
$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = 0$$

$$x_1 = 0, \qquad y_1 = 0, \qquad \theta_1 \neq 0$$



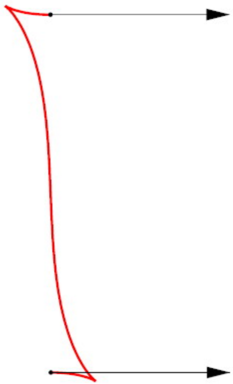
Optimal trajectories

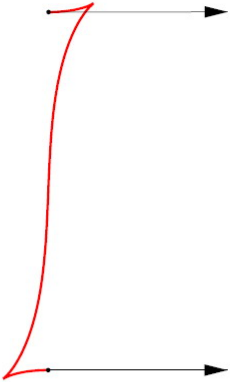
$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = \pi$$



Optimal trajectories

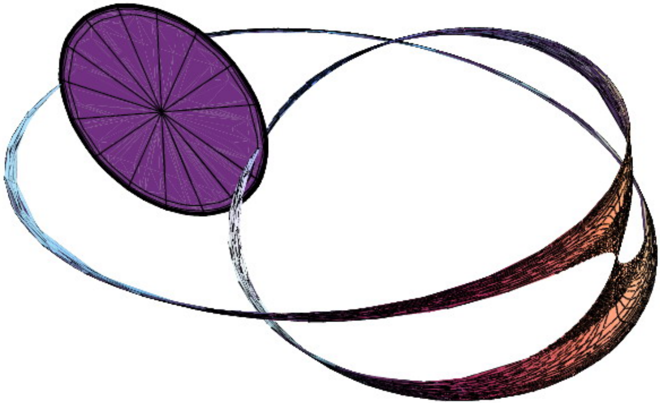
$$x_1 = 0, \qquad y_1 \neq 0, \qquad \theta_1 = 0$$





Maxwell set

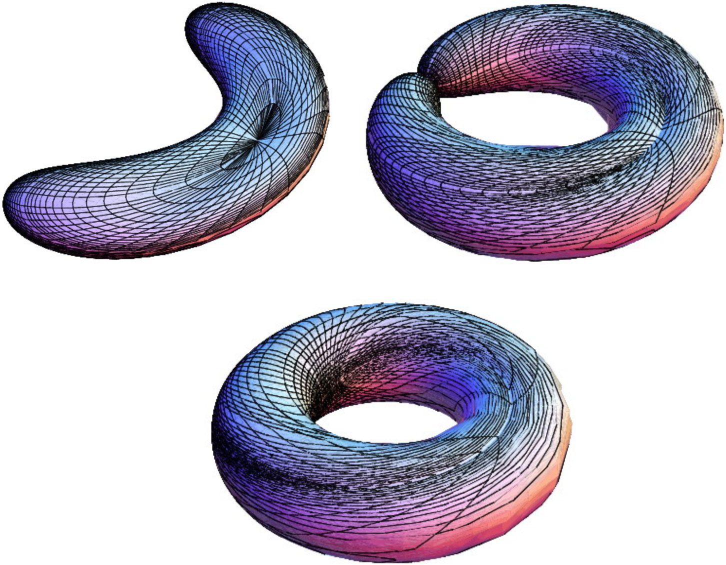
$$\mathrm{Max} = \{q_1 \in G \mid \exists > 1 \text{ optimal trajectories } q(\cdot) \; : \; q(t_1) = q_1\}$$



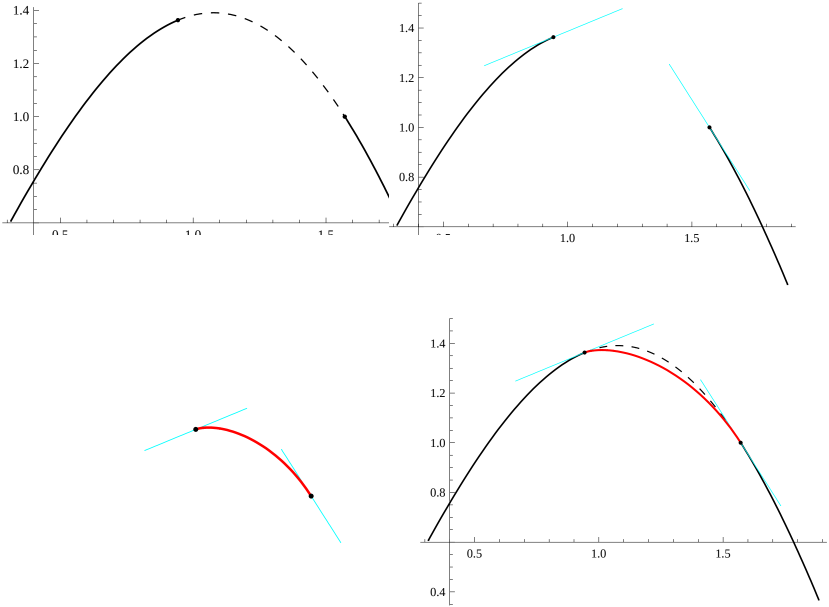
Sub-Riemannian metric and spheres

- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0,\ q(t_1) = q_1\}$
- $S_R = \{q \in G \mid d(q_0, q) = R\}$
- $R = 0 \quad \Rightarrow \quad S_R = \{q_0\}$
- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2$
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\}$
- $R > \pi \quad \Rightarrow \quad S_R \cong \mathbb{T}^2$

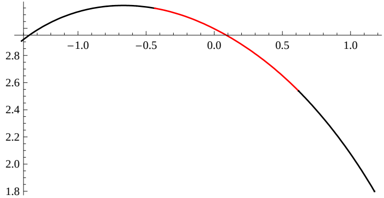
Global structure of sub-Riemannian spheres in SE(2)

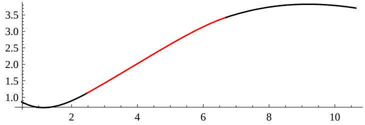


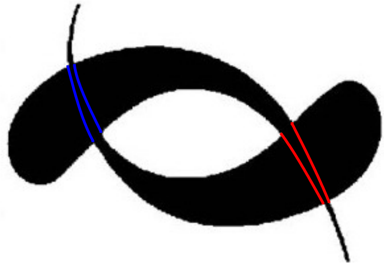
Reconstruction of corrupted curve

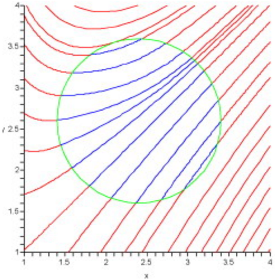


Reconstruction of corrupted curve

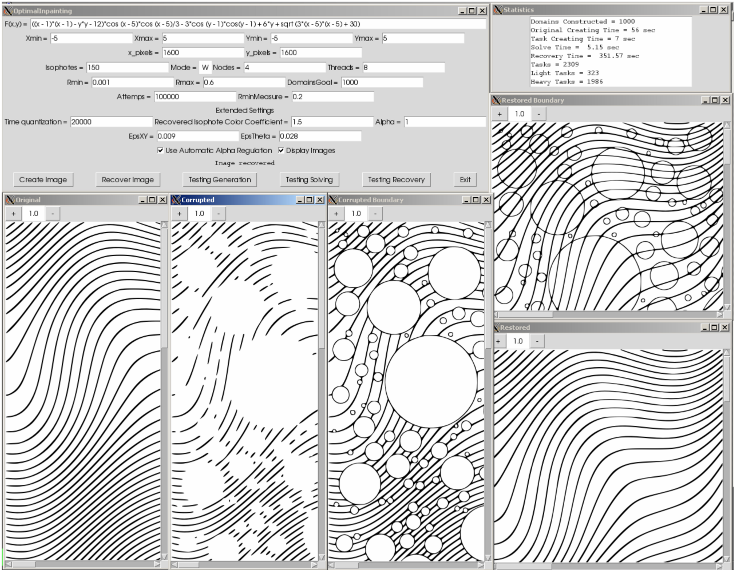




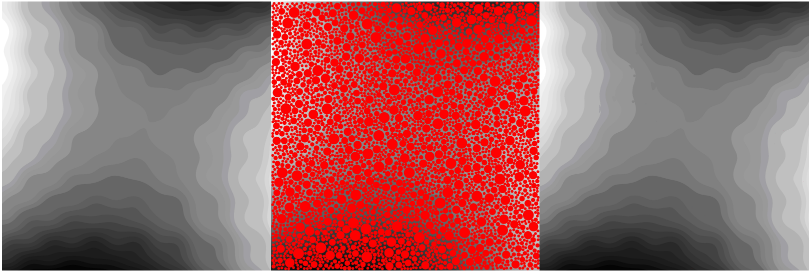




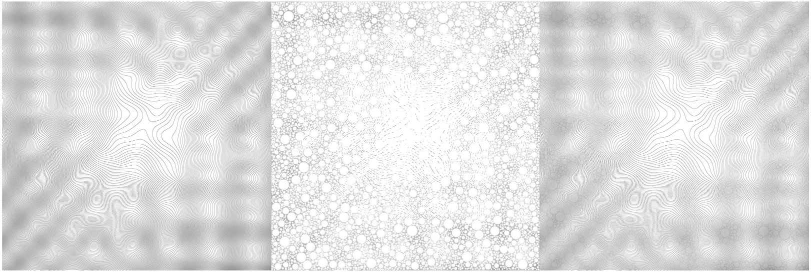
Parallel software tOptimalInpainting for corrupted images reconstruction



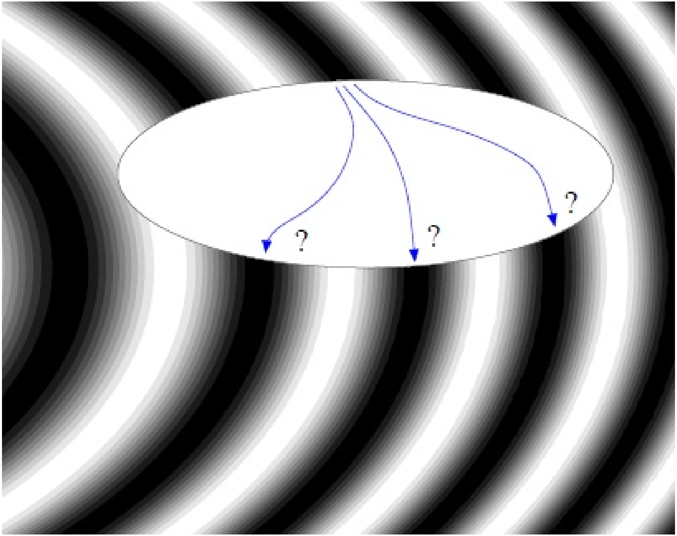
Original, corrupted, and reconstructed grayscale image



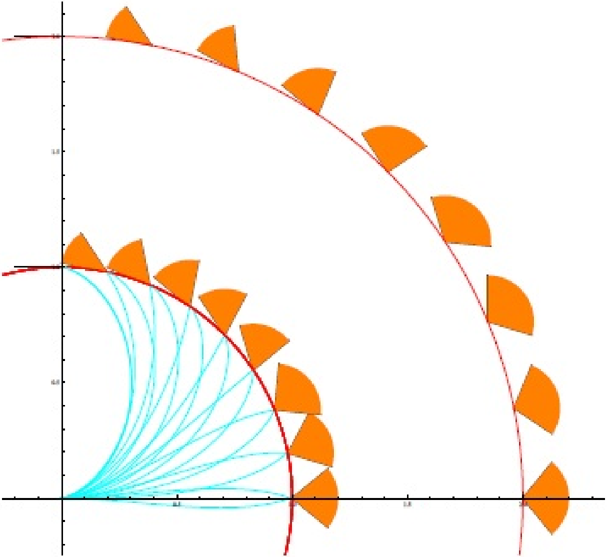
Original, corrupted, and reconstructed binary image



Problem 1: non-uniqueness of level sets



Problem 2: cusp points



Problem 3: extraction of isophotes (level lines of brightness) from half-tone image

Problem 4: critical points of brightness

Image inpainting via hypoellpitic diffusion on $PT\mathbb{R}^2$

Corrupted image $I : D \setminus \Omega \to [0, +\infty)$

1. Smoothing $f = I * G_\sigma$:
   $f(x, y) = \iint_{\mathbb{R}^2} I(\tilde{x}, \tilde{y}) G_\sigma(x - \tilde{x}, y - \tilde{y})\, d\tilde{x}\, d\tilde{y}$,
   $G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$.
2. Lift $f : \mathbb{R}^2 \to \mathbb{R}$ to $\overline{f} : PT\mathbb{R}^2 \to \mathbb{R}$:
   $$\overline{f}(x, y, \theta) = \begin{cases} f(x, y), & \text{if } \theta \text{ is the slope of } \{f = \text{const}\}, \\ 0 \text{ else} \end{cases}$$
3. Hypoelliptic (anisotropic) diffusion
   $$\partial_t \Phi(q, t) = (X_1^2 + X_2^2)\Phi(q, t),$$
   $$\Phi(q, 0) = \overline{f}(q)$$
4. Projection to $\mathbb{R}^2$:
   $$\tilde{f}(x, y) = \max_{\theta \in P^1} \Phi(x, y, \theta, T).$$

Isotropic diffusion in $\mathbb{R}^3$

- Heat equation

$$\begin{aligned} &\partial_t \Phi(x,y,z,t) = (\partial_x^2 + \partial_y^2 + \partial_z^2)\Phi(x,y,z,t),\\ &\Phi(x,y,z,0) = \varphi(x,y,z). \end{aligned}$$

- Fundamental solution (heat kernel)

$$\begin{aligned} &\partial_t \mathcal{E}(x,y,z,t) - (\partial_x^2 + \partial_y^2 + \partial_z^2)\mathcal{E}(x,y,z,t) = \delta(x,y,z,t),\\ &\mathcal{E}(x,y,z,t) = \frac{\theta(t)}{(2\sqrt{\pi t})^3} \exp\left(-\frac{x^2+y^2+z^2}{4t}\right),\\ &\Phi = \varphi * \mathcal{E}. \end{aligned}$$

- Propagation of isotropic diffusion along Riemannian geodeics

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1\partial_x + u_2\partial_y + u_3\partial_z,$$

$$\int \sqrt{u_1^2 + u_2^2 + u_3^2}\, dt \to \min .$$

Anisotropic diffusion in $PT\mathbb{R}^2$

- PDE of anisotropic diffusion

$$\partial_t \Phi(q,t) = (X_1^2 + X_2^2)\Phi(q,t),$$
$$\Phi(q,0) = \varphi(q), \qquad q = (x,y,\theta) \in PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1.$$

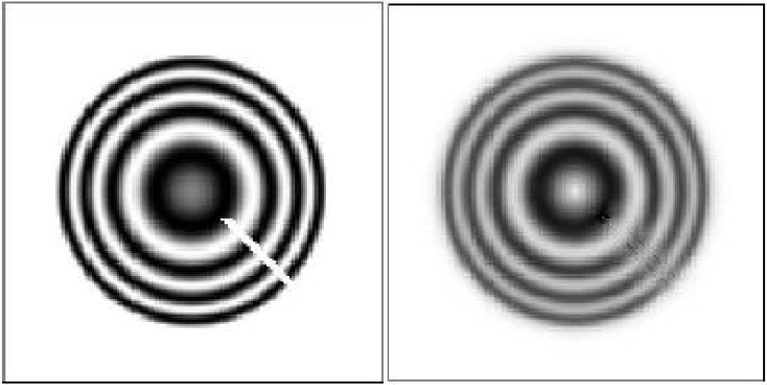
- Fundamental solution (kernel of anisotropic diffusion)

$$\partial_t \mathcal{E}(q,t) - (X_1^2 + X_2^2)\mathcal{E}(q,t) = \delta(q,t),$$
$$\mathcal{E}(q,t) = \dots,$$
$$\Phi = \varphi * \mathcal{E}.$$

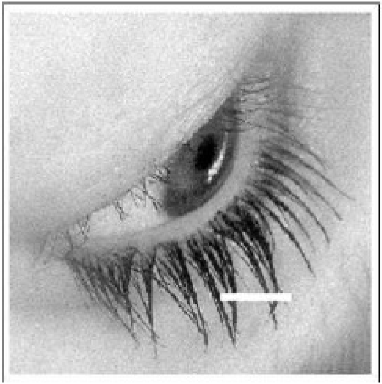
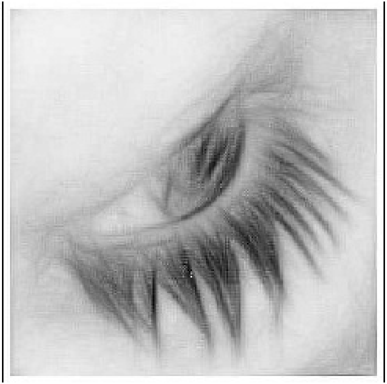
- Propagation of anisotropic diffusion along sub-Riemannian geodeics

$$\dot q = u_1 X_1(q) + u_2 X_2(q),$$
$$\int \sqrt{u_1^2 + u_2^2}\, dt \to \min.$$

Anisotropic diffusion: experiments



Anisotropic diffusion: experiments

Anisotropic diffusion: experiments