# Sub-Riemannian geometry on Carnot groups: Rank 2 and beyond

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# Part 0
# Introduction to sub-Riemannian geometry

1987 A.M. Vershik, V.Ya. Gershkovich, Nonholonomic Dynamical Systems. Geometry of distributions and variational problems.

1996 A. Bellaïche, The tangent space in sub-Riemannian geometry

1996 M. Gromov, Carnot-Carathéodory spaces seen from within

1997 V. Jurdjevic, Geometric Control Theory

2002 R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications.

2004 A.A. Agrachev, Yu. S., Control theory from the geometric viewpoint

# Sub-Riemannian minimizers and geodesics

- SR geometry $(M, \Delta, g)$:
  - smooth manifold $M$,
  - vector distribution $\Delta \subset TM$,
  - inner product $g$ in $\Delta$.
- Horizontal curve $q(\cdot)$:
  $q \in Lip([0, t_1], M)$, $\dot{q}(t) \in \Delta_{q(t)}$ for almost all $t \in [0, t_1]$.
- Length of $q(\cdot)$:

$$l = \int_0^{t_1} \sqrt{g(\dot{q}(t), \dot{q}(t))}\, dt.$$

- SR distance between $q_0, \ q_1 \in M$:

$$d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ horizontal}, \ q(0) = q_0, \ q(t_1) = q_1\}.$$

- Minimizer $q(\cdot)$:
  horizontal curve such that $l(q(\cdot)) = d(q(0), q(t_1))$.
- Geodesic $q(\cdot)$:
  horizontal curve whose small subarcs are minimizers.

# 3D Heisenberg group

- $M = \mathbb{R}^3$, $\Delta = \mathrm{span}(X_1, X_2)$, $g(X_i, X_j) = \delta_{ij}$,
- $X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $\quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$.
- Left-invariant SR geometry $\quad \Rightarrow \quad q_0 = \mathrm{Id} = (0, 0, 0)$.
- Geodesics: $q_{\theta,c}(t)$, $\theta \in S^1$, $c \in \mathbb{R}$, $t \in [0, +\infty)$,

$$x = (\sin(\theta + ct) - \sin\theta)/c,$$
$$y = (\cos\theta - \cos(\theta + ct))/c,$$
$$z = (ct - \sin(ct))/(2c^2).$$

- Minimizers: $q_{\theta,c}(t)$, $\theta \in S^1$, $c \in \mathbb{R}$, $t \in [0, 2\pi/|c|]$.
- Distance between $q_0$ and $q_1 = (x_1, y_1, z_1)$:

  $d(q_0, q_1) = \dfrac{\tau}{\sin\tau} r_1$, $r_1 = \sqrt{x_1^2 + y_1^2}$,

  $\tau \in [0, \pi]$, $\dfrac{2\tau - \sin 2\tau}{\sin^2\tau} = \dfrac{8|z_1|}{r_1^2}$.

# Optimal control problem for SR minimizers

- $\Delta = \operatorname{span}(X_1, \dots, X_k), \quad g(X_i, X_j) = \delta_{ij}$

$$\dot{q}(t) \in \Delta_{q(t)} \iff \dot{q}(t) = \sum_{i=1}^{k} u_i(t) X_i(q(t)),$$

$$l = \int_0^{t_1} \sqrt{g(\dot{q}, \dot{q})}\, dt = \int_0^{t_1} \sqrt{\sum_{i=1}^{k} u_i^2}\, dt \to \min$$

- Existence of a horizontal curve with $q(0) = q_0$, $q_{t_1} = q_1$: by Chow-Rashevsky Theorem: iff

$$\operatorname{span}(X_i(q),\ [X_i, X_j](q),\ \dots) = T_q M \ \forall q \in M.$$

- Existence of a minimizer with $q(0) = q_0$, $q_{t_1} = q_1$: by Filippov's theorem if one of the conditions hold:
    - $q_1$ close to $q_0$,
    - SR balls $B(q_0, R)$ are compact,
    - $\Delta$, $g$ left-invariant on a Lie group $M$.

# Hamiltonian vector field and its integrability

- Pontryagin maximum principle:
  $q(\cdot)$ geodesic " $\Rightarrow$ " it has a Hamiltonian lift $\lambda(t) \in T^*_{q(t)}M$ such that

$$\dot{\lambda} = \vec{H}(\lambda), \qquad \lambda \in T^*M,$$

$$H = \frac{1}{2}\sum_{i=1}^{k} h_i^2, \qquad h_i(\lambda) = \langle \lambda, X_i(q) \rangle$$

  (modulo abnormal geodesics).

- Liouville integrability of the Hamiltonian vector field $\vec{H}$:
  Find $n$ independent integrals $h_1, \ldots, h_n$ in involution:

$$\{H, h_i\} = 0, \ \{h_i, h_j\} = 0, \ d_\lambda h_1, \ldots d_\lambda h_n \text{ independent a.e. on } T^*M$$

- Complicated for general SR structures $\Rightarrow$ approximation?

# Linear approximation?

Let $k = \dim \Delta_q = 2$.

$$\dot{x} = u_1 X_1(x_0) + u_2 X_2(x_0), \qquad x \in \mathbb{R}^n$$

- $n = 2 \quad \Rightarrow \quad$ Euclidean geometry approximates Riemannian one
- $n > 2 \quad \Rightarrow \quad$ no controllability
- $\mathrm{Lie}(X_1(x_0), X_2(x_0))$ Abelian
- more sophisticated Lie algebras?

# Nilpotent approximation

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in G \simeq \mathbb{R}^n$$

- $\mathrm{Lie}(X_1, X_2)$ nilpotent:

$$[X_{i_{r+1}}, \dots, [X_{i_2}, X_{i_1}] \dots] = 0, \qquad i_j \in \{1, 2\},$$

  $r$ — step of Lie algebra $\mathrm{Lie}(X_1, X_2)$.
- $G$ Lie group
- $X_i$ left-invariant vector fields on $G$
- $G$ Carnot group: connected, simply connected, with graded Lie algebra

$$L = \oplus_{i=1}^{r} L_i, \ [L_1, L_i] = L_{i+1}, \ [L_i, L_j] \subset L_{i+j}, \ L_1 = \mathrm{span}\{X_1, X_2\}.$$

- Fundamental local approximation of generic SR structure
- E.g. left-invariant SR geometry on the 3D Heisenberg group approximates general contact 3D SR geometry (A.A.Agrachev, J.-P.Gauthier)

# Rank 2 Carnot sub-Riemannian geometry

- Growth vectors considered: $(2,3,4)$, $(2,3,5)$, $(2,3,5,8)$.
- Problems studied:
  - existence and smoothness of SR geodesics,
  - integrability of the normal Hamiltonian system of PMP,
  - regular and chaotic dynamics of the Hamiltonian system,
  - effective parametrization of exponential mapping,
  - discrete and continuous symmetries,
  - fixed points of the group of symmetries,
  - resulting study of global and local optimality of SR geodesics,
  - optimal synthesis (terminal point $\mapsto$ optimal trajectory),
  - applications in robotics, mechanics, robotics,
  - effective software, including parallel for supercomputers[1]

[1] see my affiliation :(((

# Rank 2 Carnot sub-Riemannian geometry

- Matryoshka of growth vectors:

$$(2,3,4) \subset (2,3,5) \subset (2,3,5,8).$$

- Problems studied:
  - existence and smoothness of SR geodesics,
  - integrability of the normal Hamiltonian system of PMP,
  - regular and chaotic dynamics of the Hamiltonian system,
  - effective parametrization of exponential mapping,
  - discrete and continuous symmetries,
  - fixed points of the group of symmetries,
  - resulting study of global and local optimality of SR geodesics,
  - optimal synthesis (terminal point $\mapsto$ optimal trajectory),
  - applications in robotics, mechanics, robotics,
  - effective software, including parallel for supercomputers.

# Part I

# Growth vector (2,3,4)

# Nilpotent SR geometry on the Engel group

Originals:

1. A.Ardentov, Yu.S., Extremal trajectories in nilpotent sub-Riemannian problem on Engel group, *Sbornik Mathematics*, 202 (2011), No. 11, 31–54. English translation: *Sbornik: Mathematics* (2011), 202(11):1593–1616.
2. A.Ardentov, Yu.S., Conjugate points in nilpotent sub-Riemannian problem on the Engel group, *Journal of Mathematical Sciences*, Vol. 195, No. 3, December, 2013, 369–390. `arXiv:1209.2865v1 [math.OC] 13 Sep 2012`
3. A.Ardentov, Yu.S., Cut time in nilpotent sub-Riemannian problem on the Engel group, *in preparation*

See the next talk by Andrei Ardentov

# Part II

# Growth vector (2,3,5)

# Nilpotent SR geometry on the Cartan group

Originals:


1. Exponential mapping in the generalized Dido problem (in Russian), *Matem. Sbornik*, **194** (2003), 9: 63–90. English translation: *Sbornik: Mathematics* (2003), 194(9):1331–1360.
2. Symmetries of Flat Rank Two Distributions and Sub-Riemannian Structures, *Transactions of the American Mathematical Society*, **356** (2004), 2: 457–494.
3. Discrete symmetries in generalized Dido's problem, *Sbornik: Mathematics*, 197 (2006), 2: 235–257.
4. The Maxwell set in the generalized Dido problem, *Sbornik: Mathematics*, 197 (2006), 4: 595–621.
5. Complete description of the Maxwell strata in the generalized Dido problem, *Sbornik: Mathematics*, 197 (2006), 6: 901–950.

# Results and plans for the nilpotent $(2,3,5)$ SR problem

Results

- Sub-Riemannian geodesics were parametrized.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- Software for computing length minimizers for given boundary points was developed.
- Software for solving motion planning problem via nilpotent approximation was developed for general $(2,3,5)$ systems, and tested for $S^2$ rolling on $\mathbb{R}^2$ without slipping or twisting, and a car with 2 off-hooked trailers.

Plans

- Prove that the cut time is equal to the first Maxwell time.
- Describe the optimal synthesis.
- Describe the global structure of cut locus.
- Study singularities of SR sphere for the nilpotent and general $(2,3,5)$ SR structures.
- Consider other $(2,3,5)$ SR geometries.

# Part III
# Growth vector (2,3,5,8)
# Nilpotent SR geometry
# on the free nilpotent Carnot group
# with 2 generators, of step 4

Originals:

1. Sub-Riemannian geodesics on the free Carnot group with the growth vector $(2, 3, 5, 8)$, *submitted*, `arXiv:1404.7752v1 [math.OC] 30 Apr 2014`.
2. E.Sachkova, Yu.S., Abnormal geodesics in $(2, 3, 5, 8)$ sub-Riemannian geometry, *work in progress*
3. I. Beschastnyj, L. Lokutsievsky, Yu.S., Regular and chaotic dynamics in $(2, 3, 5, 8)$ sub-Riemannian geometry, *work in progress*

# Free nilpotent sub-Riemannian problem of rank 2, step $r$

$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, \dots, X_n)$
$n = \sum_{k=1}^{r} \ell_2(k)$, $k\ell_2(k) = 2^k - \sum_{j|k} j\ell_d(j)$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in G, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Normal Hamiltonian system of Pontryagin Maximum Principle:
$h_i(\lambda) = \langle \lambda, X_i \rangle, \qquad \lambda \in T^*G,$
$H(\lambda) = \frac{1}{2}(h_1^2(\lambda) + h_2^2(\lambda))$
Is $\dot{\lambda} = \vec{H}(\lambda)$ Liouville integrable?

# Integrals of the Hamiltonian system

Symmetries:

- right translations in $G$
  $Y_1, \dots, Y_n \in \mathrm{Vec}(G)$, $(R_g)_* Y_i = Y_i$
- Rotations in $\Delta = \mathrm{span}(X_1, X_2)$:
  $Z \in \mathrm{so}(\Delta)$

Poisson algebra of integrals of the Hamiltonian field $\vec{H}$:

$$I = \mathrm{span}(H;\, g_1, \dots, g_n;\, f;\, C_1, \dots C_k)$$

- System Hamiltonian $H$,
- Right-invariant Hamiltonians $g_i(\lambda) = \langle \lambda, Y_i \rangle$,
- Hamiltonian of rotation: $f(\lambda) = \langle \lambda, Z \rangle$,
- Casimir functions. $C_j : L^* \to \mathbb{R}$, $\{C_j, h_i\} = 0 \ \forall\, i = 1, \dots, n$

1) Is $\vec{H}$ Liouville integrable?
2) Does $I$ contain $n$ independent commuting integrals?

# Growth vector $(2, 3)$

$$L = \mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, X_2, X_3)$$
$$[X_1, X_2] = X_3$$
$$G = \mathbb{R}^3_{xyz}$$
$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial z}$$

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \begin{cases} \dot{h}_1 = -h_2 h_3 \\ \dot{h}_2 = h_1 h_3 \\ \dot{h}_3 = 0 \\ \dot{q} = h_1 X_1 + h_2 X_2 \end{cases}$$

Integrability in trigonometric functions

# Liouville integrability in the $(2,3)$ case

- Right-invariant frame $Y_1, Y_2, Y_3 \in \mathrm{Vec}(G)$, $Y_i(\mathrm{Id}) = -X_i(\mathrm{Id})$:
  $Y_1 = -\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $Y_2 = -\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$, $Y_3 = -\frac{\partial}{\partial z}$
- Rotation $X_0 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

Integrals of the field $\vec{H}$:

- System Hamiltonian $H = (h_1^2 + h_2^2)/2$
- right-invariant Hamiltonians
  $g_1 = -h_1 - yh_3$, $g_2 = -h_2 + xh_3$, $g_3 = -h_3$
- Hamiltonian of rotations $h_0(\lambda) = \langle \lambda, X_0 \rangle$
- Casimir function $h_3$

Algebra of integrals: $I = \mathrm{span}(H,\, g_1, g_2, g_3,\, h_0)$

Abelian subalgebra: $\widetilde{I} = \mathrm{span}(H,\, g_2, g_3)$

# Growth vector $(2, 3, 5)$

$$L = \text{Lie}(X_1, X_2) = \text{span}(X_1, X_2, X_3, X_4, X_5)$$

$$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_2, X_3] = X_5$$

$$G = \mathbb{R}^5_{xyzvw}$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2+y^2}{2}\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v},$$
$$X_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial v} + y\frac{\partial}{\partial w}, \qquad X_4 = \frac{\partial}{\partial v}, \qquad X_5 = \frac{\partial}{\partial w}$$

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \begin{cases} \dot{h}_1 = -h_2 h_3 \\ \dot{h}_2 = h_1 h_3 \\ \dot{h}_3 = h_1 h_4 + h_2 h_5 \\ \dot{h}_4 = \dot{h}_5 = 0 \\ \dot{q} = h_1 X_1 + h_2 X_2 \end{cases}$$

# Integrability of the field $\vec{H}$ in the case $(2,3,5)$

$2H = h_1^2 + h_2^2 = r^2 = \mathrm{const}$

$h_1 = r\cos\theta$, $h_2 = r\sin\theta$, $h_3 = c$, $h_4 = \alpha\sin\beta$, $h_5 = -\alpha\cos\beta$

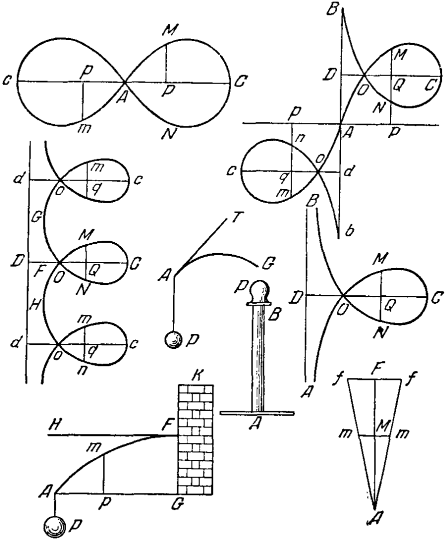
$\dot{\theta} = c$, $\dot{c} = -\alpha r\sin(\theta - \beta)$, $\dot{\alpha} = \dot{\beta} = 0$

$E = c^2/2 - \alpha r\cos(\theta - \beta)$ energy of pendulum

Integrability in Jacobi's functions

$(x(t), y(t))$ Euler elasticae

# Description of elasticae by Leonhard Euler (1744)



# Integrals of the field $\vec{H}$ in the $(2, 3, 5)$ case

- System Hamiltonian $H = (h_1^2 + h_2^2)/2$
- Right-invariant Hamiltonians $g_1$, $g_2$, $g_3$, $g_4 = -h_4$, $g_5 = -h_5$
- Hamiltonian of rotation $h_0(\lambda)$
- Casimir functions $E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$, $h_4$, $h_5$

Algebra of integrals: $I = \mathrm{span}(H,\ g_1, \ldots, g_5,\ h_0,\ E)$

Abelian subalgebra: $\widetilde{I} = \mathrm{span}(H,\ g_3, g_4, g_5,\ E)$

# Growth vector $(2, 3, 5, 8)$

$L = \mathrm{Lie}(X_1, X_2) = \mathrm{span}(X_1, \ldots, X_8)$

$[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$, $[X_1, X_4] = X_6$,
$[X_1, X_5] = [X_2, X_4] = X_7$, $[X_2, X_5] = X_8$.

$G = \mathbb{R}^8_{x_1 \ldots x_8}$

Realization of nilpotent Lie algebras by polynomial vector fields:
M.Grayson, R.Grossman, Nilpotent Lie algebras and vector fields (1989)

# Left-invariant frame in the case $(2,3,5,8)$

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8},$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2x_2}{4}\frac{\partial}{\partial x_7},$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + \frac{x_1^2}{2}\frac{\partial}{\partial x_6} + x_1x_2\frac{\partial}{\partial x_7} + \frac{x_2^2}{2}\frac{\partial}{\partial x_8},$$

$$X_4 = \frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_6} + x_2\frac{\partial}{\partial x_7},$$

$$X_5 = \frac{\partial}{\partial x_5} + x_1\frac{\partial}{\partial x_7} + x_2\frac{\partial}{\partial x_8},$$

$$X_6 = \frac{\partial}{\partial x_6}, \qquad X_7 = \frac{\partial}{\partial x_7}, \qquad X_8 = \frac{\partial}{\partial x_8}.$$

# Abnormal extremals

$$h_1 = h_2 = h_3 = 0,$$

$$\begin{pmatrix} \dot{h}_4 \\ \dot{h}_5 \end{pmatrix} = A \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}, \quad A = \begin{pmatrix} h_7 & -h_6 \\ h_8 & -h_7 \end{pmatrix} = \mathrm{const}$$

$$\dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0,$$

$$\dot{q} = -h_5 X_1 + h_4 X_2.$$

Abnormal extremals are smooth

- $\det A < 0 \quad \Rightarrow \quad (x_1(t), x_2(t))$ hyperbolas,
- $\det A > 0 \quad \Rightarrow \quad (x_1(t), x_2(t))$ ellipses,
- $\det A = 0 \quad \Rightarrow \quad (x_1(t), x_2(t))$ parabolas or lines.

Hyperbolas and parabolas are not normal trajectories $\quad \Rightarrow \quad$ they are strictly abnormal

# Normal Hamiltonian vector field in the case $(2, 3, 5, 8)$

$$\dot{\lambda} = \vec{H}(\lambda) \quad : \quad \begin{cases} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = h_1 h_3, \\ \dot{h}_3 = h_1 h_4 + h_2 h_5, \\ \dot{h}_4 = h_1 h_6 + h_2 h_7, \\ \dot{h}_5 = h_1 h_7 + h_2 h_8, \\ \dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0, \\ \dot{q} = h_1 X_1(q) + h_2 X_2(q) \end{cases}$$

# Integrals of the field $\vec{H}$ in the case $(2, 3, 5, 8)$

- System Hamiltonian $H = (h_1^2 + h_2^2)/2$
- Right-invariant Hamiltonians $g_1$, $g_2$, $g_3$, $g_4$, $g_5$, $g_6 = -h_6$, $g_7 = -h_7$, $g_8 = -h_8$
- Hamiltonian of rotation $h_0$
- Casimir functions
  $C = h_5^2 h_6 - 2h_4 h_5 h_7 + h_4^2 h_8 - 2h_3(h_6 h_8 - h_7^2) = C(g_3, \dots, g_8)$, $h_6$, $h_7$, $h_8$

Algebra of integrals: $I = \operatorname{span}(H,\, g_1, \dots g_8,\, h_0)$

Abelian subalgebra: $\widetilde{I} = \operatorname{span}(H,\, g_3, \dots g_8)$

# Homogeneous integrals of $\vec{H}$ in the case $(2, 3, 5, 8)$

Theorem

*Let a homogeneous polynomial* $P_k = P_k(h_1, h_2)$, $\deg P_k = k$, *be an integral of the field* $\vec{H}$. *Then:*

1. $P_1 = \sum_{i=6}^{8} a_i h_i$,
2. $P_2 = \sum_{i,j=6}^{8} a_{ij} h_i h_j + bH$,
3. $P_3 = \sum_{i,j,k=6}^{8} a_{ijk} h_i h_j h_k + \sum_{i=6}^{8} b_i h_i H + aC$.

- Homogeneous integrals of order $k \leq 3$ are expressed through Casimir functions $h_6$, $h_7$, $h_8$, $C$ and system Hamiltonian $H$.
- Are there nontrivial homogeneous integrals of order $k > 3$?
- 7 independent integrals in involution: $H$, $g_3$, $\ldots$ $g_8$.
- Is $\vec{H}$ Liouville integrable?

# Reduction of the vertical (adjoint) subsystem for the system $\dot{\lambda} = \vec{H}(\lambda)$

$2H = h_1^2 + h_2^2 = 1$

$h_1 = \cos\theta,\ h_2 = \sin\theta,$

$h_3 = c,\ h_4 = a,\ h_5 = b,\ h_6 = m,\ h_7 = p,\ h_8 = n$

$$\begin{aligned}
\dot{\theta} &= c,\\
\dot{c} &= a\cos\theta + b\sin\theta,\\
\dot{a} &= m\cos\theta + p\sin\theta,\\
\dot{b} &= p\cos\theta + n\sin\theta,\\
m,\ n,\ p &= \mathrm{const}\,.
\end{aligned}$$

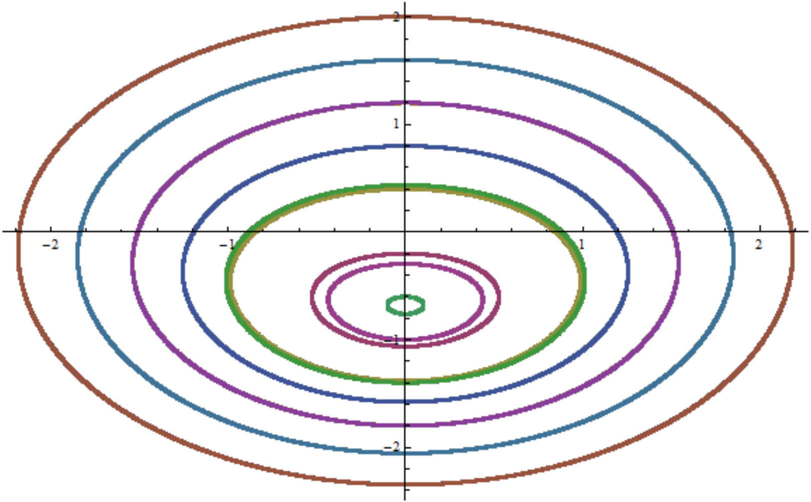
$\delta = p^2 - mn \neq 0 \quad \Rightarrow$

$$\begin{aligned}
\dot{\theta} &= (2pab - na^2 - mb^2)/(2\delta) + k,\\
\dot{a} &= m\cos\theta + p\sin\theta,\\
\dot{b} &= p\cos\theta + n\sin\theta, \qquad m,\ n,\ p,\ k = \mathrm{const}.
\end{aligned}$$

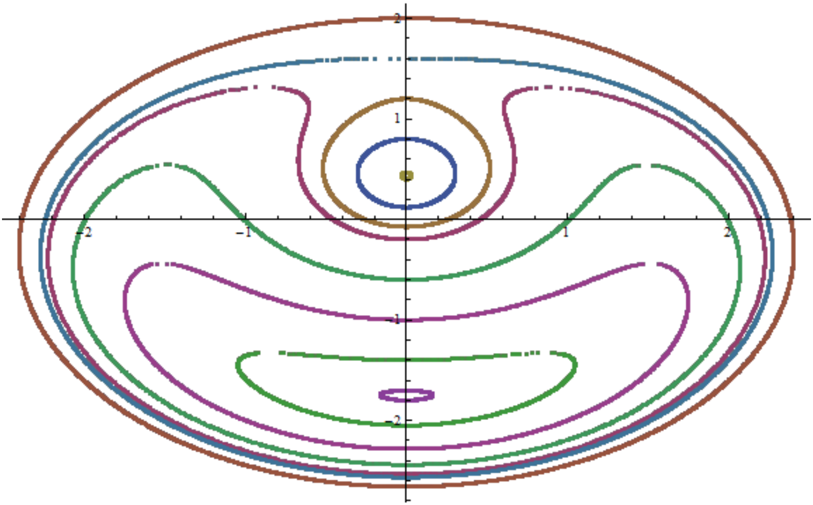
# Conclusion

- Free nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5, 8)$,
- Realization by polynomial vector fields in $\mathbb{R}^8$,
- Symmetries of the Hamiltonian systems,
- Integrals of the normal Hamiltonian system:
  10 integrals, of which commute only 7,
- Casimir functions and co-adjoint orbits computed,
- 1-st, 2-nd, and 3-rd order (w.r.t. $h_i$) integrals computed
- Mishchenko-Fomenko's noncommutative integrability theory fails
- (Non) commutative integrability remains an open question.
- Numerical evidence of Liouville non-integrability

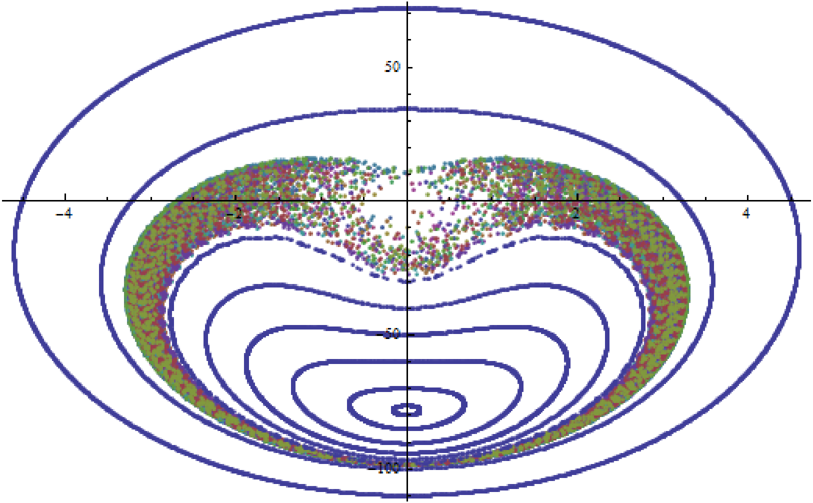
# Orbits of Poincaré mapping: Integrable case with 1 periodic trajectory



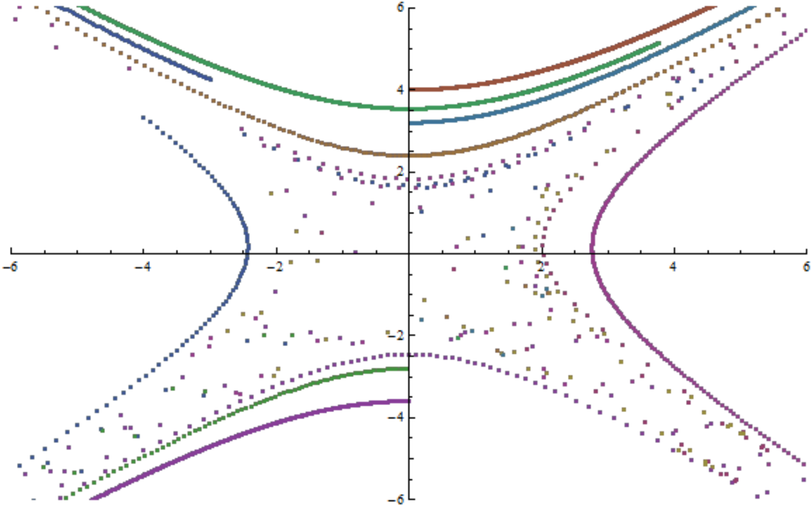
# Orbits of Poincaré mapping: Integrable case with 2 periodic trajectories



# Orbits of Poincaré mapping: Elliptic non-integrable case



# Orbits of Poincaré mapping: Hyperbolic non-integrable case



## Two-parametric family of open problems:

rank $k$ free nilpotent sub-Riemannian problems of step $N$.