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CONTROLLABILITY OF RIGHT-INVARIANT SYSTEMS ON SOLVABLE LIE GROUPS

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ABSTRACT. We study controllability of right-invariant control systems $\Gamma = A + \mathbb{R}B$ on Lie groups. Necessary and sufficient controllability conditions for Lie groups not coinciding with their derived subgroup are obtained in terms of the root decomposition corresponding to the adjoint operator $\operatorname{ad} B$. As an application, right-invariant systems on metabelian groups and matrix groups, and bilinear systems are considered.

1. Introduction

Control systems with a Lie group as a state space are studied in the mathematical control theory since the early 1970-ies.

R. W. Brockett [1] considered applied problems leading to control systems on matrix groups and their homogeneous spaces; e.g., a model of DC to DC conversion and the rigid body control raise control problems on the group of rotations of the three-space $\mathrm{SO}(3)$ and on the group $\mathrm{SO}(3) \times \mathbb{R}^3$ respectively. The natural framework for such problems are *matrix control systems* of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) B_i x(t), \quad u_i(t) \in \mathbb{R}, \tag{1}$$

where $x(t)$ and A, $B_1, \dots, B_m$ are $n \times n$ matrices. There was established the basic *rank controllability test* for homogeneous ($A = 0$) systems: such systems are controllable iff the Lie algebra generated by the matrices $B_1, \dots, B_m$ has the full dimension. This test was specified for the groups of matrices with positive determinant $\mathrm{GL}_+(n, \mathbb{R})$, the group of matrices with

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determinant one $\mathrm{SL}(n, \mathbb{R})$, the group of symplectic matrices $\mathrm{Sp}(n)$, and the group of orthogonal matrices with determinant one $\mathrm{SO}(n)$. Some controllability conditions for nonhomogeneous matrix systems were also obtained.

The first systematic mathematical study of control systems on Lie groups was fulfilled by V. Jurdjevic and H. J. Sussmann [2]. They noticed that the passage from the matrix system (1) to the more general *right-invariant system*

$$\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} u_i(t) B_i(x(t)), \quad x(t) \in G, \quad u(t) \in \mathbb{R}, \tag{2}$$

where A, $B_1, \ldots, B_m$ are right-invariant vectorfields on a Lie group G, "in no essential way affects the nature of the problem." The basic properties of the attainable set (the semi-group property, path-connectedness, relation with the associated Lie subalgebras determined by the vectorfields A, $B_1, \ldots, B_m$) were established. The rank controllability test was proved for system (2) in the homogeneous case and in the case of a compact group G. Sufficient controllability conditions for other cases were also given.

V. Jurdjevic and I. Kupka [4] introduced a systematic tool for studying controllability on Lie groups. For the control system (2) presented in the form of the polysystem

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\} \subset L \tag{3}$$

(where L is the Lie algebra of the group G) they considered its *Lie saturation* $\mathrm{LS}(\Gamma)$ — the largest system equivalent to Γ. Controllability of the system Γ on G is equivalent to $\mathrm{LS}(\Gamma) = L$, and a general technique for verification of this equality was proposed. (This technique is outlined in Subsec. 4.2 and used in Subsecs. 4.3, 4.4 below.) In [4] sufficient controllability conditions for the single-input systems $\Gamma = \{ A + uB \}$ were obtained for simple and semi-simple groups G with the use of this technique. They were given in terms of the root decomposition of the algebra L corresponding to the adjoint operator $\operatorname{ad} B$.

In their preceding paper V. Jurdjevic and I. Kupka [3] presented the enlargement technique for systems on matrix groups $G \subset \mathrm{GL}(n, \mathbb{R})$ and obtained sufficient controllability conditions for $G = \mathrm{SL}(n, \mathbb{R})$ and $G = \mathrm{GL}_+(n, \mathbb{R})$.

These results for $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{GL}_+(n, \mathbb{R})$ were generalized by J.P. Gauthier and G. Bornard [5].

B. Bonnard, V. Jurdjevic, I. Kupka, and G. Sallet [6] obtained a characterization of controllability on a Lie group which is a semidirect product of a vector space and a compact group which acts linearly on the vector space. The case $G = \mathbb{R}^n \otimes_s \mathrm{SO}(n)$ was applied to the study of Serret–Frenet moving frames.

The results of [4] for simple and semi-simple Lie groups were generalized in a series of papers by J.P. Gauthier, I. Kupka, and G. Sallet [7], R. El Assoudi and J. P. Gauthier [9], [10], F. Silva Leite and P.E. Crouch [8]: analogous controllability conditions were obtained for classical Lie groups with the use of the Lie saturation technique and the known structure of real simple and semi-simple Lie algebras.

In contrast to this "*simple*" progress, invariant systems on *solvable* groups seem not to be studied in the geometric control theory at all until 1993. Then a complete solution of the controllability problem for simply connected nilpotent groups G was given by V. Ayala Bravo and L. San Martin [11]. Some results on controllability of (not right-invariant) systems on Lie groups analogous to linear systems on $\mathbb{R}^n$ were obtained by V. Ayala Bravo and J. Tirao [12].

Several results on controllability of right-invariant systems were obtained within the framework of the Lie semigroups theory [13], [14]: for nilpotent groups by J. Hilgert, K.H. Hofmann, and J. D. Lawson [15], for reductive groups by J. Hilgert [16]. For Lie groups G with cocompact radical, J. D. Lawson [17] proved that controllability of a system $\Gamma \subset L$ follows from nonexistence of a half-space in L bounded by a Lie subalgebra and containing Γ; if G is additionally simply connected, this condition is also necessary for controllability. This result generalizes controllability conditions for compact groups [2], nilpotent groups [15], and for semidirect products of vector groups and compact groups [6].

In [18] the author characterized controllability of hypersurface right-invariant systems, i.e., of systems Γ of the form (3) with the codimension one Lie subalgebra generated by the vectorfields $B_1, \ldots, B_m$. This gave a necessary controllability condition for simply connected groups — the *hypersurface principle*, see its formulation for single-input systems Γ in Proposition 2 below. In its turn, the hypersurface principle was applied and there was obtained a controllability test for simply connected solvable Lie groups G with Lie algebra L satisfying the additional condition: for all $X \in L$ the adjoint operator $\operatorname{ad} X$ has real spectrum.

The aim of this paper is to give convenient controllability conditions of single-input systems Γ for a wide class of Lie groups including solvable ones; more precisely, for Lie groups not coinciding with their derived subgroups.

The structure of this paper is as follows.

We state the problem and introduce the notation in Sec. 2.

In Sec. 3 we give the necessary controllability condition for simply connected groups G not coinciding with their derived subgroup $G^{(1)}$ (Theorem 1 and Corollary 1). These propositions are proved in Subsec. 3.3 after the preparatory work in Subsec. 3.2. The main tools are the rank controllability condition (Proposition 1) and the hypersurface principle (Proposition 2).

Sec. 4 is devoted to sufficient controllability conditions for the groups $G \neq G^{(1)}$. We present the main sufficient results in Subsec. 4.1. Then we recall the Lie saturation technique in Subsec. 4.2 and prove preliminary lemmas in Subsec. 4.3. The main results (Theorem 2 and Corollaries 2, 3) are proved in Subsec. 4.4.

In Sec. 5 we consider several applications of our results. Controllability conditions for metabelian groups are obtained in Subsec. 5.1. Then controllability conditions for some subgroup of the group of motions of the Euclidean space are studied in detail (Subsec. 5.2) and are applied to bilinear systems (Subsec. 5.3). Finally, the clear small-dimensional version of this theory for the group of motions of the two-dimensional plane is presented in Subsec 5.4.

A preliminary version of the below results was stated in [19].

2. Problem statement and definitions

Let G be a connected Lie group, L its Lie algebra (i.e., the Lie algebra of right-invariant vector fields on G), and A, B any elements of L. The single-input affine *right-invariant control system* on G is a subset of L of the form

$$\Gamma = \{ A + uB \mid u \in \mathbb{R} \}.$$

The *attainable set* $\mathbf{A}$ of the system Γ is the subsemigroup of G generated by the set of the one-parameter semigroups

$$\{ \exp(tX) \mid X \in \Gamma, \, t \in \mathbb{R}_+ \}.$$

The system Γ is called *controllable* if $\mathbf{A} = G$.

To see the relation of these notions with the standard system-theoretical ones, let us write the right-invariant vector fields A and B as $A(x)$ and $B(x)$, $x \in G$. Then the system Γ can be written in the customary form

$$\dot{x}(t) = A(x(t)) + u(t)B(x(t)), \quad u(t) \in \mathbb{R}, \quad x(t) \in G.$$

The attainable set $\mathbf{A}$ is then the set of points of the state space G reachable from the identity element of the group G for any nonnegative time. The system Γ is controllable iff any point of G can be reached along trajectories of this system from the identity element of the group G. By right-invariance of the fields $A(x)$, $B(x)$, the identity element in the previous sentence can be changed by an arbitrary one.

Our aim is to characterize controllability of the system Γ in terms of the Lie group G and the right-invariant vector fields A and B.

Now we introduce the notation we will use in the sequel.

For any subset $l \subset L$ we denote by $\mathrm{Lie}\,(l)$ the Lie subalgebra of L generated by l. Closure of a set M is denoted by $\mathrm{cl}\, M$. The signs $\oplus$ and

$\sum^{\oplus}$ denote direct sums of vector spaces; $\oplus_{\rm s}$ and $\otimes_{\rm s}$ stand for semidirect products of Lie algebras and Lie groups correspondingly.

We denote by Id the identity operator or the identity matrix of appropriate dimension,

$$\mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\alpha(t) = \begin{pmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{pmatrix}, \quad M_{r,\alpha} = \begin{pmatrix} r & -\alpha \\ \alpha & r \end{pmatrix}$$

for t, α, $r \in \mathbb{R}$. The square matrix with all zero entries except one unit in the ith raw and the jth column is denoted by E_{ij}.

Now we introduce the notation connected with eigenvalues and eigenspaces of the adjoint operator $\operatorname{ad} B$ in L:

- the derived subalgebra and the second derived subalgebra:

$$L^{(1)} = [L, L], \qquad L^{(2)} = [L^{(1)}, L^{(1)}],$$

- the complexifications of L and $L^{(i)}$, $i = 1, 2$:

$$L_c = L \otimes \mathbb{C}, \qquad L_c^{(i)} = L^{(i)} \otimes \mathbb{C}$$

(the tensor products over $\mathbb{R}$),

- the adjoint representations and operators:

$$\operatorname{ad} : L \to \operatorname{End}(L), \qquad (\operatorname{ad} B)X = [B, X] \quad \forall X \in L,$$

$$\operatorname{ad}_c : L_c \to \operatorname{End}(L_c), \qquad (\operatorname{ad}_c B)X = [B, X] \quad \forall X \in L_c,$$

- spectra of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$\operatorname{Sp}^{(i)} = \left\{ a \in \mathbb{C} \mid \operatorname{Ker}(\operatorname{ad}_c B|_{L_c^{(i)}} - a \operatorname{Id}) \neq \{0\} \right\},$$

- real and complex eigenvalues of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$\operatorname{Sp}_r^{(i)} = \operatorname{Sp}^{(i)} \cap \mathbb{R}, \qquad \operatorname{Sp}_c^{(i)} = \operatorname{Sp}^{(i)} \setminus \mathbb{R},$$

- complex eigenspaces of $\operatorname{ad}_c B|_{L_c^{(1)}}$:

$$L_c(a) = \operatorname{Ker}(\operatorname{ad}_c B|_{L_c^{(1)}} - a \operatorname{Id}),$$

- real eigenspaces of $\operatorname{ad} B|_{L^{(1)}}$:

$$L(a) = (L_c(a) + L_c(\overline{a})) \cap L,$$

- complex root subspaces of $\operatorname{ad}_c B|_{L_c^{(i)}}$, $i = 1, 2$:

$$L_c^{(i)}(a) = \cup_{N=1}^{\infty} \operatorname{Ker}(\operatorname{ad}_c B|_{L_c^{(i)}} - a \operatorname{Id})^N,$$

- real root subspaces of $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$L^{(i)}(a) = \left(L_c^{(i)}(a) + L_c^{(i)}(\overline{a})\right) \cap L,$$

• real components of $L^{(i)}$, $i = 1, 2$:

$$L_r^{(i)} = \sum\nolimits^{\oplus} \{\, L^{(i)}(a) \mid a \in \mathrm{Sp}_r^{(i)},\ \operatorname{Im} a \geq 0 \,\}.$$

Note that the subalgebras $L^{(1)}$ and $L^{(2)}$ are ideals of L, so they are $(\operatorname{ad} B)$-invariant, and the restrictions $\operatorname{ad} B|_{L^{(1)}}$ and $\operatorname{ad} B|_{L^{(2)}}$ are well defined.

In the following lemma we collect several simple statements about decomposition of the subalgebras $L^{(1)}$ and $L^{(2)}$ into sums of root spaces and eigenspaces of the adjoint operator $\operatorname{ad} B$.

Lemma 2.1.

(1) $L^{(i)} = \sum^{\oplus} \{\, L^{(i)}(a) \mid a \in \mathrm{Sp}^{(i)}, \operatorname{Im} a \geq 0 \,\}$, $i = 1, 2$,
(2) $\mathrm{Sp}^{(2)} \subset \mathrm{Sp}^{(1)}$, $\mathrm{Sp}_r^{(2)} \subset \mathrm{Sp}_r^{(1)}$,
(3) $L^{(2)}(a) \subset L^{(1)}(a)$ *for any* $a \in \mathrm{Sp}^{(2)}$,
(4) $L_r^{(2)} \subset L_r^{(1)}$,
(5) $\mathrm{Sp}^{(2)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$.

Proof. Is obtained by the standard linear-algebraic arguments. In item (5) Jacobi's identity is additionally used. $\square$

Consider the quotient operator

$$\widetilde{\operatorname{ad} B} \ : \ L^{(1)}/L^{(2)} \to L^{(1)}/L^{(2)}$$

defined as follows:

$$(\widetilde{\operatorname{ad} B})(X + L^{(2)}) = (\operatorname{ad} B)X + L^{(2)} \quad \forall X \in L^{(1)}.$$

Analogously for $a \in \mathrm{Sp}^{(1)}$ we define the quotient operator in the quotient root space:

$$\widetilde{\operatorname{ad} B(a)} \ : \ L^{(1)}(a)/L^{(2)}(a) \to L^{(1)}(a)/L^{(2)}(a),$$
$$(\widetilde{\operatorname{ad} B(a)})(X + L^{(2)}(a)) = (\operatorname{ad} B)X + L^{(2)}(a) \quad \forall X \in L^{(1)}(a),$$

and its complexification:

$$\widetilde{\operatorname{ad}_c B(a)} \ : \ L_c^{(1)}(a)/L_c^{(2)}(a) \to L_c^{(1)}(a)/L_c^{(2)}(a),$$
$$(\widetilde{\operatorname{ad}_c B(a)})(X + L_c^{(2)}(a)) = (\operatorname{ad}_c B)X + L_c^{(2)}(a) \quad \forall X \in L_c^{(1)}(a).$$

Definition 1. Let $a \in \mathrm{Sp}^{(1)}$. We denote by $\mathrm{j}\,(a)$ the geometric multiplicity of the eigenvalue a of the operator $\widetilde{\operatorname{ad}_c B(a)}$ in the vector space $L_c^{(1)}(a)/L_c^{(2)}(a)$.

Remarks.

(a) For $a \in \mathrm{Sp}^{(1)}$ the number $\mathrm{j}\,(a)$ is equal to the number of Jordan blocks of the operator $\widetilde{\mathrm{ad}\,B(a)}$ in the space $L^{(1)}(a)/L^{(2)}(a)$.

(b) If an eigenvalue $a \in \mathrm{Sp}^{(1)}$ is simple, then $\mathrm{j}\,(a) = 0$ for $a \in \mathrm{Sp}^{(2)}$ and $\mathrm{j}\,(a) = 1$ for $a \in \mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)}$.

Suppose that $L = L^{(1)} \oplus \mathbb{R}B$ (this assumption will be justified by Theorem 1 below). Then by Lemma 2.1

$$L = \mathbb{R}B \oplus L^{(1)} = \mathbb{R}B \oplus \sum\nolimits^{\oplus} \{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Im}\, a \geq 0 \,\},$$

that is why any element $X \in L$ can uniquely be decomposed as follows:

$$X = X_B + \sum \{\, X(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Im}\, a \geq 0 \,\}, \quad X_B \in \mathbb{R}B,\ X(a) \in L^{(1)}(a).$$

We will consider such decomposition for the uncontrolled vector field A of the system Γ:

$$A = A_B + \sum \{\, A(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Im}\, a \geq 0 \,\}.$$

We denote by $\widetilde{A(a)}$ the canonical projection of the vector $A(a) \in L^{(1)}(a)$ onto the quotient space $L^{(1)}(a)/L^{(2)}(a)$.

Definition 2. Let $L = L^{(1)} \oplus \mathbb{R}B$, $a \in \mathrm{Sp}^{(1)}$, and $\mathrm{j}\,(a) = 1$. We say that a vector A *has the zero a-top* if

$$\widetilde{A(a)} \in (\widetilde{\mathrm{ad}\,B(a)} - a\,\mathrm{Id})(L^{(1)}(a)/L^{(2)}(a)).$$

In the opposite case we say that A *has a nonzero a-top*. We use the corresponding notations: $\mathrm{top}\,(A, a) = 0$ or $\mathrm{top}\,(A, a) \neq 0$.

Remark. Geometrically, if a vector A has a nonzero a-top, then the vector $\widetilde{A(a)}$ has a nonzero component corresponding to the highest adjoined vector in the (single) Jordan chain of the operator $\widetilde{\mathrm{ad}\,B(a)}$. Due to nonuniqueness of the Jordan base, this component is nonuniquely determined, but its property to be zero is basis-independent.

Definition 3. A pair of complex numbers (α, β), $\mathrm{Re}\,\alpha \leq \mathrm{Re}\,\beta$, is called an *$N$-pair of eigenvalues of the operator* $\mathrm{ad}\,B$ if the following conditions hold:

(1) $\alpha, \beta \in \mathrm{Sp}^{(1)}$,

(2) $L^{(2)}(\alpha) \not\subset \sum \{\, [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)},$
$\mathrm{Re}\, a,\ \mathrm{Re}\, b \notin [\mathrm{Re}\,\alpha; \mathrm{Re}\,\beta] \,\},$

(3) $L^{(2)}(\beta) \not\subset \sum \{\, [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)},$
$\mathrm{Re}\, a,\ \mathrm{Re}\, b \notin [\mathrm{Re}\,\alpha; \mathrm{Re}\,\beta] \,\}.$

Remarks.

(a) In other words, to generate the both root spaces $L^{(2)}(\alpha)$ and $L^{(2)}(\beta)$ for an N-pair (α, β), we need at least one root space $L^{(1)}(\gamma)$ with $\operatorname{Re}\gamma \in [\alpha; \beta]$. The name is explained by the fact that N-pairs can NOT be overcome by the extension process described in Lemma 4.2: they are the strongest obstacle to controllability under the necessary conditions of Theorem 1.

(b) The property of absence of the real N-pairs will be used to formulate sufficient controllability conditions in Theorem 2. In some generic cases this property can be verified by Lemma 4.3.

3. Necessary controllability conditions

§ 3.1. Main theorem and known results. It turns out that controllability on simply connected Lie groups G with $G \neq G^{(1)}$ is a very strong property: it imposes many restrictions both on the group G and on the system Γ.

Theorem 1. *Let a Lie group G be simply connected and its Lie algebra L satisfy the condition $L \neq L^{(1)}$. If a system Γ is controllable, then*:

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $L_r^{(2)} = L_r^{(1)}$,
(4) $\operatorname{Sp}_r^{(2)} = \operatorname{Sp}_r^{(1)}$,
(5) $\operatorname{Sp}^{(1)}{}_r \subset \operatorname{Sp}^{(1)} + \operatorname{Sp}^{(1)}$,
(6) $\mathrm{j}\,(a) \leq 1$ *for all* $a \in \operatorname{Sp}^{(1)}$,
(7) $\operatorname{top}(A, a) \neq 0$ *for all* $a \in \operatorname{Sp}^{(1)}$ *for which* $\mathrm{j}\,(a) = 1$.

The notations $\mathrm{j}\,(a)$ and $\operatorname{top}(A, a)$ used in Theorem 1 are explained in Definitions 1 and 2 in Sec. 2.

Remarks.

(a) The first condition is a characterization of the state space G but not of the system Γ. It means that no single-input system $\Gamma = \{ A + uB \}$ can be controllable on a simply connected Lie group G with $\dim G^{(1)} < \dim G - 1$. That is, to control on such a group, one has to increase the number of inputs. There is a general lower estimate $m > \dim G - \dim G^{(1)}$ for the number of the controlled vectorfields $B_1, \ldots, B_m$ necessary for controllability of the multi-input system (3) on a simply connected group G [18].

(b) Conditions (3)–(7) are nontrivial only for Lie algebras L with $L^{(2)} \neq L^{(1)}$ (in particular, for solvable noncommutative L). If $L^{(2)} = L^{(1)}$, then these conditions are obviously satisfied.

(c) The third condition means that $\mathrm{j}\,(a) = 0$ for all $a \in \mathrm{Sp}_r^{(1)}$, that is why condition (6) is nontrivial only for $a \in \mathrm{Sp}_c^{(1)}$.

(d) By the same reason, in condition 7 the inclusion $a \in \mathrm{Sp}^{(1)}$ can be changed by $a \in \mathrm{Sp}_c^{(1)}$. Note that if $\mathrm{j}\,(a) = 0$, then by the formal Definition 2 the vector A has the zero a-top.

(e) The fourth and fifth conditions are implied by the third one but are easier to verify. The simple (and strong) "arithmetic" necessary controllability condition (5) can be verified by a single glance at spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$.

(f) For solvable L under conditions (1), (2) the spectrum

$$\mathrm{Sp}^{(1)} = \mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}})$$

is the same for all $B \notin L^{(1)}$ modulo homotheties. Then conditions (3)–(5) depend on L but not on B.

(g) For the case of simple spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$ the necessary controllability conditions take respectively the more simple form:

Corollary 1. *Let a Lie group G be simply connected and its Lie algebra L satisfy the condition $L \neq L^{(1)}$. Suppose that the spectrum $\mathrm{Sp}^{(1)}$ is simple. If a system Γ is controllable, then*:

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$,
(4) $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$,
(5) $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)}$.

Theorem 1 and Corollary 1 will be proved in Subsec. 3.3.

Remark. Now we discuss the condition $L^{(1)} \neq L$ essential for this work and motivated by its initial focus — solvable Lie algebras L. Consider a Levi decomposition

$$L = R \oplus_{\mathrm{s}} S, \quad R = \mathrm{rad}\, L.$$

It is well known (see, e.g., [20], Theorem 3.14.1) that the Levi decomposition of the derived subalgebra is then

$$L^{(1)} = [L, R] \oplus_{\mathrm{s}} S, \quad [L, R] = \mathrm{rad}\, L^{(1)}.$$

This means that

$$L^{(1)} \neq L \iff [L, \mathrm{rad}\, L] \neq \mathrm{rad}\, L.$$

If a Lie algebra L is semisimple (i.e., $\mathrm{rad}\, L = 0$), then obviously $L^{(1)} = L$. The converse is generally not true (although this is asserted by [21], Sec. 87, Corollary 3). For example, for the Lie algebra $\mathbb{R}^3 \oplus_{\mathrm{s}} \mathrm{so}(3)$ (which is the Lie algebra of the Lie group of motions of the three-space) its derived subalgebra coincides with the algebra itself. (This example was kindly indicated to the author by A. A. Agrachev).

The main tools to obtain the necessary controllability conditions given in Theorem 1 is the rank controllability condition and the hypersurface principle.

The system Γ is said to satisfy the rank controllability condition if the Lie algebra generated by Γ coincides with L:

$$\operatorname{Lie}(\Gamma) = \operatorname{Lie}(A, B) = L.$$

Proposition 1. (Theorem 7.1, [2]). *The rank controllability condition is necessary for controllability of a system Γ on a group G.*

Generally, the attainable set $\mathbf{A}$ lies (and has a nonempty interior, which is dense in $\mathbf{A}$) in the connected subgroup of G corresponding to the Lie algebra $\operatorname{Lie}(A, B)$.

The hypersurface principle is formulated for the system Γ as follows:

Proposition 2. (Corollary 3.2, [18]). *Let a Lie group G be simply connected, $A, B \in L$, and let the Lie algebra L have a codimension one subalgebra containing B. Then the system $\Gamma = A + \mathbb{R}B$ is not controllable on G.*

The sense of this proposition is that under the hypotheses stated there exists a codimension one subgroup of the group G which separates G into two disjoint parts, is tangent to the field B, and is intersected by the field A in one direction only. Then the attainable set $\mathbf{A}$ lies "to one side" of this subgroup.

Notice that the property of absence of a codimension one subalgebra of L containing B is sufficient for controllability of Γ on a Lie group G with cocompact radical; if G is additionally simply connected, this condition is also sufficient (Corollary 12.6, [17]).

§ 3.2. Preliminary lemmas. First we obtain several conditions sufficient for existence of codimension one subalgebras of a Lie algebra L containing a vector $B \in L$.

Lemma 3.1. *Suppose that $L^{(1)} + \mathbb{R}B \neq L$. Then there exists a codimension one subalgebra of L containing B.*

Proof. Denote by l the vector space $L^{(1)} + \mathbb{R}B$. We have $[l, l] \subset L^{(1)} \subset l$, that is why l is a subalgebra; any vector space containing l is a subalgebra of L too. Since $l \neq L$, there exists a codimension one subspace l_1 of L containing l. Then l_1 is the required codimension one subalgebra of L containing B. $\square$

Lemma 3.2. *Let $L^{(1)} \oplus \mathbb{R}B = L$. If $L_r^{(2)} \neq L_r^{(1)}$, then there exists a codimension one subalgebra of L containing B.*

Proof. If $L_r^{(1)} \neq L_r^{(2)}$, then there exists a real eigenvalue $a_0 \in \operatorname{Sp}_r^{(1)}$ such that $L^{(1)}(a_0) \neq L^{(2)}(a_0)$. Let

$$\{v_1 + L^{(2)}(a_0), \dots, v_p + L^{(2)}(a_0)\}, \quad p = \dim(L^{(1)}(a_0)/L^{(2)}(a_0)),$$

be a Jordan base of the operator $\widetilde{\operatorname{ad} B(a_0)}$:

$$\widetilde{\operatorname{ad} B(a_0)}(v_i + L^{(2)}(a_0)) = (a_0 v_i + v_{i+1}) + L^{(2)}(a_0), \ i = 1, \dots, p-1, \tag{4}$$

$$\widetilde{\operatorname{ad} B(a_0)}(v_p + L^{(2)}(a_0)) = a_0 v_p + L^{(2)}(a_0). \tag{5}$$

(We suppose, for simplicity, that the eigenvalue a_0 of the operator $\widetilde{\operatorname{ad} B(a_0)}$ is geometrically simple, i.e., matrix of this operator is a single Jordan block; for the general case of several Jordan blocks the changes of the proof are obvious.)

Consider the vector space

$$l_1 = \operatorname{span}(v_2, \dots, v_p) \oplus L^{(2)}(a_0).$$

It follows from (4), (5) that the space l_1 is $(\operatorname{ad} B)$-invariant. Additionally, we have $\dim l_1 = \dim L^{(1)}(a_0) - 1$.

Then we define the vector spaces

$$\begin{aligned} l_2 &= l_1 \oplus \sum\nolimits^{\oplus} \{ L^{(1)}(a) \mid a \in \operatorname{Sp}^{(1)}, \, a \neq a_0, \, \operatorname{Im} a \geq 0 \}, \\ l_3 &= l_2 \oplus \mathbb{R}B. \end{aligned} \tag{6}$$

First, $\dim l_2 = \dim L^{(1)} - 1$, that is why

$$\dim l_3 = \dim L^{(1)} = \dim L - 1. \tag{7}$$

Second, $L^{(1)}(a_0) \supset l_1 \supset L^{(2)}(a_0)$, hence,

$$L^{(1)} \supset l_2 \supset L^{(2)}. \tag{8}$$

Third, the space l_2 is $(\operatorname{ad} B)$-invariant. That is why, by virtue of (6) and (8), we obtain the chain

$$[l_3, l_3] = [l_2, B] + [l_2, l_2] \subset l_2 + [L^{(1)}, L^{(1)}] = l_2 + L^{(2)} = l_2 \subset l_3.$$

Hence, l_3 is the required subalgebra of L: it has codimension one (see (7)) and contains the vector B (see (6)). $\square$

In the following three lemmas we obtain conditions sufficient for violation of the rank controllability condition, i.e., necessary for controllability.

Lemma 3.3. *Suppose that $B \notin L^{(1)}$ and let there exist a vector subspace $l_1 \subset L$ such that the following relations hold*:

(1) $L^{(2)} \subset l_1 \subset L^{(1)}$,
(2) $l_1 \neq L^{(1)}$,
(3) $A \in \mathbb{R}B \oplus l_1$,
(4) $(\operatorname{ad} B) l_1 \subset l_1$.

Then $\operatorname{Lie}(A, B) \neq L$.

Proof. By condition (1),

$$[l_1, l_1] \subset [L^{(1)}, L^{(1)}] = L^{(2)} \subset l_1,$$

i.e., l_1 is a Lie subalgebra.

Consider the vector space $l = \mathbb{R}B \oplus l_1$. We have (in view of condition (4))

$$[l, l] \subset (\operatorname{ad} B) l_1 + [l_1, l_1] \subset l_1 \subset l,$$

so l is a Lie subalgebra too.

By condition (3) we have $\operatorname{Lie}(A, B) \subset l$, and condition (2) implies $l \neq L$. Hence, $\operatorname{Lie}(A, B) \neq L$. $\square$

Lemma 3.4. *Let $L = \mathbb{R}B \oplus L^{(1)}$. If $\mathrm{j}(a_0) > 1$ for some $a_0 \in \mathrm{Sp}^{(1)}$, then* $\operatorname{Lie}(A, B) \neq L$.

Proof. Consider the case of the complex $a_0 \in \mathrm{Sp}_c^{(1)}$ first. By the condition $\mathrm{j}(a_0) > 1$, the quotient operator $\widetilde{\operatorname{ad}_c B(a_0)}$ has at least two cyclic spaces $V, W \subset L_c^{(1)}(a_0)/L_c^{(2)}(a_0)$. That is, there are two Jordan chains $\{v_1, \dots, v_p\}$, $\{w_1, \dots, w_q\}$, $p, q > 0$, such that

$$V = \operatorname{span}(v_1, \dots, v_p), \quad W = \operatorname{span}(w_1, \dots, w_q),$$

and in these bases matrices of the operators $\widetilde{\operatorname{ad}_c B(a_0)}|_V$ and $\widetilde{\operatorname{ad}_c B(a_0)}|_W$ are the Jordan blocks

$$\underbrace{\begin{pmatrix} a_0 & 0 & \cdots & 0 & 0 \\ 1 & a_0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & a_0 & 0 \\ 0 & 0 & \cdots & 1 & a_0 \end{pmatrix}}_{p}, \qquad \underbrace{\begin{pmatrix} a_0 & 0 & \cdots & 0 & 0 \\ 1 & a_0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & a_0 & 0 \\ 0 & 0 & \cdots & 1 & a_0 \end{pmatrix}}_{q}. \tag{9}$$

(Obviously, we can assume that the complex conjugate bases $\{\overline{v}_1, \dots, \overline{v}_p\}$ and $\{\overline{w}_1, \dots, \overline{w}_q\}$ form Jordan chains of the operator $\widetilde{\operatorname{ad}_c B(\overline{a}_0)}$ in the complex conjugate spaces $\overline{V}, \overline{W} \subset L_c^{(1)}(\overline{a}_0)/L_c^{(2)}(\overline{a}_0)$.)

Notice that

$$(\operatorname{ad}_c \widetilde{B(a_0)}) \operatorname{span}(v_i, \dots, v_p) \subset \operatorname{span}(v_i, \dots, v_p), \qquad i = 1, \dots, p, \tag{10}$$

$$(\operatorname{ad}_c \widetilde{B(a_0)}) \operatorname{span}(w_j, \dots, w_q) \subset \operatorname{span}(w_j, \dots, w_q), \quad j = 1, \dots, q. \tag{11}$$

For the direct sum

$$L_c^{(1)}(a_0)/L_c^{(2)}(a_0) = V \oplus W \oplus (\text{other cyclic spaces of } \operatorname{ad}_c \widetilde{B(a_0)}) \tag{12}$$

consider the decomposition

$$\begin{aligned} \widetilde{A(a_0)} = (A_{v_1} v_1 + \ldots + A_{v_p} v_p) + (A_{w_1} w_1 + \ldots + A_{w_q} w_q) + \\ + \ (\text{components in other cyclic spaces of } \operatorname{ad}_c \widetilde{B(a_0)}). \end{aligned} \tag{13}$$

We can assume that

$$A_{v_1} = 0 \text{ or } A_{w_1} = 0 \text{ in decomposition (13).} \tag{14}$$

This will be proved at the end of this proof. Now suppose that condition (14) holds and, for definiteness, $A_{w_1} = 0$. That is why

$$\begin{aligned} &\widetilde{A(a_0)} \in \tilde{l} := V \oplus \operatorname{span}(w_2, \dots, w_q) \oplus (\text{other cyclic spaces of } \operatorname{ad}_c \widetilde{B(a_0)}), \\ &\quad \dim \tilde{l} = \dim(L_c^{(1)}(a_0)/L_c^{(2)}(a_0)) - 1, \end{aligned}$$

and, in view of (10), (11),

$$(\operatorname{ad}_c \widetilde{B(a_0)})\, \tilde{l} \subset \tilde{l}.$$

Now let $l \subset L_c^{(1)}(a_0)$ be the canonical preimage of the space $\tilde{l}$. Obviously,

$$\begin{gathered} A(a_0) \in l, \\ \dim l = \dim L_c^{(1)}(a_0) - 1, \\ (\operatorname{ad}_c B)\, l \subset l, \\ L_c^{(1)}(a_0) \supset l \supset L_c^{(2)}(a_0). \end{gathered}$$

Then we pass to realification:

$$\begin{gathered} A(a_0) \in l_r := (l + \overline{l}) \cap L \subset L^{(1)}(a_0), \\ \dim l_r = \dim L^{(1)}(a_0) - 2, \\ (\operatorname{ad} B)\, l_r \subset l_r, \\ L^{(1)}(a_0) \supset l_r \supset L^{(2)}(a_0). \end{gathered}$$

Finally, for the space $l_1 := l_r \oplus \sum^{\oplus} \{ L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ a \neq a_0,\ \mathrm{Im}\, a \geq 0 \}$ we obtain

$$\begin{gathered} A \in \mathbb{R} B \oplus l_1, \\ \dim l_1 = \dim L^{(1)} - 2, \\ (\operatorname{ad} B)\, l_1 \subset l_1, \\ L^{(1)} \supset l_1 \supset L^{(2)}. \end{gathered}$$

So all conditions of Lemma 3.3 are satisfied, and $\mathrm{Lie}\,(A, B) \neq L$ modulo the unproved condition (14).

To prove this condition, suppose that $A_{v_1} \neq 0$ and $A_{w_1} \neq 0$ in decomposition (13). In view of symmetry between V and W, we can assume that $p = \dim V > q = \dim W$. Define the new basis in V:

$$\tilde{v}_i = v_i + \frac{A_{w_1}}{A_{v_1}} w_i \quad \text{for} \quad 1 \leq i \leq q, \qquad \tilde{v}_i = v_i \quad \text{for} \quad q < i \leq p.$$

It is easy to see that $\{\tilde{v}_1, \dots, \tilde{v}_p\}$ is a basis of V, and $A_{w_1} = 0$ in decomposition (13) for the new basis $\{\tilde{v}_1, \dots, \tilde{v}_p\}$ and the old basis $\{w_1, \dots, w_q\}$.

Now we show that the new basis is a Jordan one.

If $1 \leq i < q$, then

$$\begin{aligned} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, \tilde{v}_i &= \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, v_i + \frac{A_{w_1}}{A_{v_1}} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, w_i = \\ &= (a_0 v_i + v_{i+1}) + \frac{A_{w_1}}{A_{v_1}} (a_0 w_i + w_{i+1}) = \\ &= a_0 \left(v_i + \frac{A_{w_1}}{A_{v_1}} w_i \right) + \left(v_{i+1} + \frac{A_{w_1}}{A_{v_1}} w_{i+1} \right) = a_0 \tilde{v}_i + \tilde{v}_{i+1}. \end{aligned}$$

If $i = q$ and $p > q$, then

$$\begin{aligned} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, \tilde{v}_q &= \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, v_q + \frac{A_{w_1}}{A_{v_1}} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, w_q = \\ &= (a_0 v_q + v_{q+1}) + \frac{A_{w_1}}{A_{v_1}} a_0 w_q = \\ &= a_0 \left(v_q + \frac{A_{w_1}}{A_{v_1}} w_q \right) + v_{q+1} = a_0 \tilde{v}_q + \tilde{v}_{q+1}. \end{aligned}$$

If $i = q$ and $p = q$, then

$$\begin{aligned} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, \tilde{v}_q &= \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, v_q + \frac{A_{w_1}}{A_{v_1}} \big(\widetilde{\operatorname{ad}_c B(a_0)}\big)\, w_q = \\ &= a_0 v_q + \frac{A_{w_1}}{A_{v_1}} a_0 w_q = a_0 \left(v_q + \frac{A_{w_1}}{A_{v_1}} w_q \right) = a_0 \tilde{v}_q. \end{aligned}$$

And if $q < i < p$, then

$$\big(\widetilde{\mathrm{ad}_c\, B(a_0)}\big)\tilde v_i = \big(\widetilde{\mathrm{ad}_c\, B(a_0)}\big) v_i = a_0 v_i + v_{i+1} = a_0 \tilde v_i + \tilde v_{i+1}.$$

Finally, if $i = p > q$, then

$$\big(\widetilde{\mathrm{ad}_c\, B(a_0)}\big)\tilde v_p = \big(\widetilde{\mathrm{ad}_c\, B(a_0)}\big) v_p = a_0 v_p = a_0 \tilde v_p.$$

So $\{\tilde v_1, \dots, \tilde v_p\}$ is a Jordan basis for $\widetilde{\mathrm{ad}_c\, B(a_0)}|_V$, and $A_{w_1} = 0$ in decomposition (13), as was claimed.

So the lemma is proved for the case of the complex eigenvalue a_0.

And if a_0 is real, then the proof is analogous and easier: there is no need in complexification and further realification. $\square$

Lemma 3.5. *Let* $L = \mathbb{R}B \oplus L^{(1)}$. *Suppose that* $\mathrm{j}\,(a_0) = 1$ *and* $\mathrm{top}\,(A, a_0) = 0$ *for some eigenvalue* $a_0 \in \mathrm{Sp}^{(1)}$. *Then* $\mathrm{Lie}\,(A, B) \neq L$.

Proof. The Jordan base of the operator $\widetilde{\mathrm{ad}_c\, B(a_0)}$ consists of one Jordan chain

$$\mathrm{span}(v_1, \dots, v_k) = L_c^{(1)}(a_0)/L_c^{(2)}(a_0),$$

and moreover, in the decomposition

$$\widetilde{A(a_0)} = A_{v_1} v_1 + \ldots + A_{v_k} v_k$$

we have $A_{v_1} = 0$ since

$$(\widetilde{\mathrm{ad}_c\, B(a)} - a_0\,\mathrm{Id})(L_c^{(1)}(a_0)/L_c^{(2)}(a_0)) = \mathrm{span}(v_2, \dots, v_k).$$

Then in the same way as in Lemma 3.4 we denote by l the preimage of the space $\mathrm{span}(v_2, \dots, v_k)$ under the projection $L_c^{(1)}(a_0) \to L_c^{(1)}(a_0)/L_c^{(2)}(a_0)$, and by $\bar l$ the complex conjugate to l in L_c. Then the space

$$l_1 = \big((l + \bar l) \cap L\big) \oplus \sum\nolimits^{\oplus} \big\{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ a \neq a_0,\ \mathrm{Im}\, a \geq 0 \,\big\}$$

satisfies all hypotheses of Lemma 3.3, that is why $\mathrm{Lie}\,(A, B) \neq L$. $\square$

§ 3.3. Proofs of the necessary controllability conditions.

Proof of Theorem 1. Suppose that the system Γ is controllable on the group G.

Items (1) and (2). If $\dim L^{(1)} \neq \dim L - 1$ or $B \in L^{(1)}$, then $L^{(1)} + \mathbb{R}B \neq L$. It follows from Lemma 3.1 and the hypersurface principle (Proposition 2) that Γ is not controllable. This contradiction proves items (1) and (2), and allows to assume below in the proof that $L^{(1)} \oplus \mathbb{R}B = L$.

Item (3). If $L_r^{(2)} \neq L_r^{(1)}$, then it follows from Lemma 3.2 and the hypersurface principle that Γ is not controllable.

Item (4) follows immediately from item (3).

Item (5). From the previous item we have $\mathrm{Sp}_r^{(1)} = \mathrm{Sp}_r^{(2)}$. But $\mathrm{Sp}_r^{(2)} \subset \mathrm{Sp}^{(2)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$ (see Lemma 2.1, (6)), consequently, $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$.

Items (6), (7) follow from Lemmas 3.4, 3.5 and from the rank controllability condition (Proposition 1). $\square$

Proof of Corollary 1. If the spectrum $\mathrm{Sp}^{(1)}$ is simple, then $L^{(1)}(a) = L(a)$ for all $a \in \mathrm{Sp}^{(1)}$, and $\dim L^{(1)}(a) = 1$ or 2 for $a \in \mathrm{Sp}_r^{(1)}$ or $\mathrm{Sp}_c^{(1)}$ respectively. Further, $L_r^{(2)} = L_r^{(1)}$ is equivalent to $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$, and $\mathrm{top}\,(A,a) \neq 0$ iff $A(a) \neq 0$, $a \in \mathrm{Sp}^{(1)}$. Now Corollary 1 follows immediately from Theorem 1. $\square$

4. Sufficient controllability conditions

§ 4.1. Main results. Under the necessary assumptions of Theorem 1, we can give wide sufficient controllability conditions. Notice that the assumption of simple connectedness can now be removed. So the below sufficient conditions are completely *algebraic*; this is in contrast with the *geometric* assumption (the finiteness of center of G) essential for the sufficient controllability conditions for simple and semi-simple Lie groups G [4].

Theorem 2. *Suppose that the following conditions are satisfied for a Lie algebra L and a system Γ:*

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $L_r^{(2)} = L_r^{(1)}$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(5) $\mathrm{top}\,(A,a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(6) *the operator* $\mathrm{ad}\, B|_{L^{(1)}}$ *has no N-pairs of real eigenvalues.*

Then the system Γ is controllable on any Lie group G with the Lie algebra L.

The notation $\mathrm{top}\,(A,a)$ and the notion of N-pair used in Theorem 2 are explained in Definitions 2 and 3 in Sec. 2.

Remarks. (a) Conditions (1)–(3) are necessary for controllability in the case of a simply connected $G \neq G^{(1)}$ (by Theorem 1).

(b) Conditions (4) and (5) are close to the necessary conditions (6) and (7) of Theorem 1 respectively. Notice that the fourth condition means that all complex eigenvalues of $\mathrm{ad}\, B|_{L^{(1)}}$ are geometrically simple.

(c) Conditions (2) and (5) are open, i.e., they are preserved under small perturbations of A and B.

(d) The most restrictive of conditions (1)–(6) is the last one. It can be shown that the smallest dimension of $L^{(1)}$ in which this condition is satisfied and preserved under small perturbations of spectrum of $\operatorname{ad} B|_{L^{(1)}}$ for solvable L is (6). This can be used to obtain a classification of controllable systems Γ on solvable Lie groups G with small-dimensional derived subgroups $G^{(1)}$.

(e) The technically complicated condition (6) can be changed by more simple and more restrictive one, and sufficient conditions can be given as in Corollary 2 below.

(f) Under the additional assumption of simplicity of the spectrum $\mathrm{Sp}^{(1)}$ the sufficient controllability conditions take the even more simple form presented in Corollary 3 below.

Corollary 2. *Suppose that the following conditions are satisfied for a Lie algebra L and a system Γ:*

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $L_r^{(2)} = L_r^{(1)}$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(5) $\operatorname{top}(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(6) $\mathrm{Sp}_r^{(1)} = \emptyset$, *or* $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$, *or* $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

Then the system Γ is controllable on any Lie group G with the Lie algebra L.

Corollary 3. *Suppose that the following conditions are satisfied for a Lie algebra L and a system Γ:*

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) *the spectrum* $\mathrm{Sp}^{(1)}$ *is simple,*
(4) $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$,
(5) $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(6) $\mathrm{Sp}_r^{(1)} = \emptyset$, *or* $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$, *or* $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

Then the system Γ is controllable on any Lie group G with the Lie algebra L.

Theorem 2 and Corollaries 2, 3 will be proved in Subsec. 4.4.

§ 4.2. Lie saturation. To prove the above sufficient conditions we use the notion of the Lie saturation of a right-invariant system introduced by V. Jurdjevic and I. Kupka. Now we recall the basic definition and properties necessary for us (see details in [4], pp. 163–165).

Given a right-invariant system $\Gamma \subset L$ on a Lie group G, its Lie saturation $\mathrm{LS}(\Gamma) \subset L$ is defined as follows:

$$\mathrm{LS}(\Gamma) = \mathrm{Lie}\,(\Gamma) \cap \left\{ X \in L \mid \exp(tX) \in \mathrm{cl}\,\mathbf{A} \quad \forall t \in \mathbb{R}_+ \right\}.$$

$\mathrm{LS}(\Gamma)$ is the largest (with respect to inclusion) system having the same closure of the attainable set as Γ.

Properties of Lie saturation:

(1) $\Gamma \subset \mathrm{LS}(\Gamma)$,
(2) $\mathrm{LS}(\Gamma)$ is a convex closed cone in L,
(3) if $\pm X$, $\pm Y \in \mathrm{LS}(\Gamma)$, then $\pm[X, Y] \in \mathrm{LS}(\Gamma)$,
(4) if $\pm X$, $Y \in \mathrm{LS}(\Gamma)$, then $\exp(s \,\mathrm{ad}\, X)Y \in \mathrm{LS}(\Gamma)$ for all $s \in \mathbb{R}$,
(5) controllability condition:

$$\text{if } \mathrm{LS}(\Gamma) = L, \text{ then } \Gamma \text{ is controllable.} \tag{15}$$

§ 4.3. Preliminary lemmas. In this section we assume that $L \neq L^{(1)}$ (this condition holds, e.g., for solvable L). In view of Theorem 1, we suppose additionally that $\dim L^{(1)} = \dim L - 1$ and $B \notin L^{(1)}$.

First we present a necessary technical lemma.

Lemma 4.1. *Let $\eta, \mu, \lambda \in \mathbb{R}$, $\lambda \neq 0$. Then*

$$\lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} (1 + \eta \cos(\mu t))\sigma_\lambda(t)\, dt = \begin{cases} 0 & \textit{for} \quad |\mu| \neq \lambda, \\ (\eta/2)\,\mathrm{Id} & \textit{for} \quad |\mu| = \lambda. \end{cases}$$

$$\lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} (1 + \eta \sin(\mu t))\sigma_\lambda(t)\, dt = \begin{cases} 0 & \textit{for} \quad |\mu| \neq \lambda, \\ \pm(\eta/2)\mathrm{J} & \textit{for} \quad \mu = \pm\lambda. \end{cases}$$

Proof. Is obtained by the direct computation. □

Now we prove the proposition that plays the central role in obtaining our sufficient controllability conditions (Theorem 2). It is analogous to item (a) of Proposition 11, [4].

Lemma 4.2. *Let $C \in \mathrm{LS}(\Gamma) \cap L^{(1)}$. Suppose that for any $a \in \mathrm{Sp}_c^{(1)}$ the following conditions hold*:

(1) $\dim L_c(a) = 1$,
(2) $\mathrm{top}\,(C, a) \neq 0$ *or* $L^{(1)}(a) \subset \mathrm{LS}(\Gamma)$.

Suppose additionally that for the number

$$r = \max\{\, \mathrm{Re}\, a \mid a \in \mathrm{Sp}^{(1)},\ C(a) \neq 0 \,\}$$

$$(\textit{or } r = \min\{\, \mathrm{Re}\, a \mid a \in \mathrm{Sp}^{(1)},\ C(a) \neq 0 \,\})$$

we have $r \notin \mathrm{Sp}^{(1)}$ *or* $C(r) = 0$. *Then*

$$\mathrm{LS}(\Gamma) \supset \sum \{ L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, a = r,\ a \neq r \}.$$

Proof. For simplicity suppose that

$$\mathrm{Sp}^{(1)} \cap \{ \mathrm{Re}\, z = r,\ z \neq r \} = \{ a, \overline{a}; b, \overline{b} \}, \tag{16}$$

where $a = r + \sqrt{-1}\,\alpha$, $b = r + \sqrt{-1}\,\beta$, $\alpha \neq \beta$, $\alpha, \beta > 0$. If there are more than two pairs of complex conjugate eigenvalues at the line $\{ \mathrm{Re}\, z = r \}$, the proof is analogous; if there are less than two pairs, then the proof is obviously simplified.

So we have

$$\begin{aligned} L^{(1)} = L^{(1)}(a) \oplus L^{(1)}(b) \oplus L^{(1)}(r) \oplus \\ \oplus \sum\nolimits^{\oplus} \{ L^{(1)}(c) \mid c \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, c < r, \mathrm{Im}\, c \geq 0 \} \end{aligned} \tag{17}$$

(notice that if $r \notin \mathrm{Sp}^{(1)}$, then $L^{(1)}(r) = \{0\}$ by definition), and, respectively,

$$C = C(a) + C(b) + \sum \{ C(c) \mid c \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, c < r, \mathrm{Im}\, c \geq 0 \}.$$

For any element $D \in L$, nonnegative function $g(t)$, and natural number p consider the limit

$$I(D, g, p) := \lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} \frac{g(t)}{e^{rt} t^{p-1}} \exp(t \,\mathrm{ad}\, B) D \, dt.$$

It follows from the properties of the cone $\mathrm{LS}(\Gamma)$ (see Subsec. 4.2) that if $D \in \mathrm{LS}(\Gamma)$ and the limit $I(D, g, p)$ exists, then $I(D, g, p) \in \mathrm{LS}(\Gamma)$. Moreover, if $v(t) \in \mathrm{LS}(\Gamma)$ for all $t \in \mathbb{R}$, then

$$I(D, g, p, v) := \lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} \frac{g(t)}{t^{p-1}} \left(\frac{\exp(t \,\mathrm{ad}\, B) D}{e^{rt}} - v(t) \right) \, dt \in \mathrm{LS}(\Gamma)$$

if the limit exists.

Introduce the notation

$$C_a(t) = \exp(t \,\mathrm{ad}\, B) C(a), \tag{18}$$

$$C_b(t) = \exp(t \,\mathrm{ad}\, B) C(b), \tag{19}$$

$$C_{<r}(t) = \exp(t \,\mathrm{ad}\, B) \sum \{ C(c) \mid c \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, c < r, \mathrm{Im}\, c \geq 0 \}. \tag{20}$$

Notice that

$$I(C,g,p) = \lim_{T\to+\infty} \frac{1}{T} \int\limits_T^{2T} \frac{g(t)}{e^{rt}t^{p-1}} (C_a(t) + C_b(t) + C_{<r}(t))\, dt.$$

For any bounded nonnegative function $g(t)$ and any $p \in \mathbb{N}$ we have

$$\frac{g(t)}{e^{rt}t^{p-1}} C_{<r}(t) = O(e^{-\varepsilon t} t^{1-p+d}), \quad t \to +\infty,$$

where $\varepsilon = \min\{\, r - \operatorname{Re} c \mid c \in \operatorname{Sp}^{(1)}, \operatorname{Re} c < r \,\} > 0$, and d is equal to the size of the maximal Jordan block of the operator $\operatorname{ad}_c B|_{L_c^{(1)}}$ corresponding to the eigenvalues $c \in \operatorname{Sp}^{(1)}$ with $\operatorname{Re} c < r$. That is why

$$\lim_{T\to+\infty} \frac{1}{T} \int\limits_T^{2T} \frac{g(t)}{e^{rt}t^{p-1}} C_{<r}(t)\, dt = 0.$$

Consequently,

$$I(C,g,p) = \lim_{T\to+\infty} \frac{1}{T} \int\limits_T^{2T} \frac{g(t)}{e^{rt}t^{p-1}} (C_a(t) + C_b(t))\, dt \in \operatorname{LS}(\Gamma) \qquad (21)$$

if the limit exists.

Now we choose the bases $\{\, x_1, y_1; \dots; x_k, y_k \,\}$ and $\{\, z_1, w_1; \dots; z_l, w_l \,\}$ in the spaces $L^{(1)}(a)$ and $L^{(1)}(b)$ in which matrices of the operators $\operatorname{ad} B|_{L^{(1)}(a)}$ and $\operatorname{ad} B|_{L^{(1)}(b)}$ are the Jordan blocks

$$\begin{pmatrix} M_{r,\alpha} & 0 & \cdots & 0 & 0 \\ \mathrm{Id} & M_{r,\alpha} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & M_{r,\alpha} & 0 \\ 0 & 0 & \cdots & \mathrm{Id} & M_{r,\alpha} \end{pmatrix},$$

$$\begin{pmatrix} M_{r,\beta} & 0 & \cdots & 0 & 0 \\ \mathrm{Id} & M_{r,\beta} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & M_{r,\beta} & 0 \\ 0 & 0 & \cdots & \mathrm{Id} & M_{r,\beta} \end{pmatrix},$$

where $a = r + \sqrt{-1}\,\alpha$, $b = r + \sqrt{-1}\,\beta$. In the bases $\{ x_1, y_1; \dots ; x_k, y_k \}$ and $\{ z_1, w_1; \dots ; z_l, w_l \}$ we have

$$C(a) = (C_{x_1}, C_{y_1}; \dots ; C_{x_k}, C_{y_k})' = (C_{X_1}; \dots ; C_{X_k})',$$
$$C(b) = (C_{z_1}, C_{w_1}; \dots ; C_{z_l}, C_{w_l})' = (C_{Z_1}; \dots ; C_{Z_l})',$$

where $X_i = (x_i, y_i)'$, $C_{X_i} = (C_{x_i}, C_{y_i})' \in \mathbb{R}^2$, $i = 1, \dots, k$, and $Z_i = (z_i, w_i)'$, $C_{W_i} = (C_{z_i}, C_{w_i})' \in \mathbb{R}^2$, $i = 1, \dots, l$ (the prime denotes transposition of vectors and matrices). Then in the base

$$\{ x_1, y_1; \dots ; x_k, y_k; z_1, w_1; \dots ; z_l, w_l \}$$

of the space $L^{(1)}(a) \oplus L^{(1)}(b)$ we have

$$\frac{C_a(t) + C_b(t)}{e^{rt}} = \begin{pmatrix} \sigma_\alpha(t) C_{X_1} \\ \sigma_\alpha(t)(tC_{X_1} + C_{X_2}) \\ \vdots \\ \sigma_\alpha(t)\left(\frac{t^{k-1}}{(k-1)!}C_{X_1} + \frac{t^{k-2}}{(k-2)!}C_{X_2} + \dots + C_{X_k}\right) \\ \sigma_\beta(t) C_{Z_1} \\ \sigma_\beta(t)(tC_{Z_1} + C_{Z_2}) \\ \vdots \\ \sigma_\beta(t)\left(\frac{t^{l-1}}{(l-1)!}C_{Z_1} + \frac{t^{l-2}}{(l-2)!}C_{Z_2} + \dots + C_{Z_l}\right) \end{pmatrix}. \tag{22}$$

Let $k > l$ (if $k \le l$, then the below argument can easily be modified).

(A) Now we show that $\operatorname{span}(x_k, y_k) \subset \mathrm{LS}(\Gamma)$.

According to the hypotheses of this lemma, we have $L^{(1)}(a) \subset \mathrm{LS}(\Gamma)$ or $\operatorname{top}(C, a) \ne 0$. If $L^{(1)}(a) \subset \mathrm{LS}(\Gamma)$, then $\operatorname{span}(x_k, y_k) \subset \mathrm{LS}(\Gamma)$. That is why we suppose below that $\operatorname{top}(C, a) \ne 0$, which means in the base $\{ x_1, y_1; \dots ; x_k, y_k \}$ that $C_{X_1} \ne 0$.

Set $g(t) = 1 + \eta\cos(\mu t)$ $(g(t) = 1 + \eta\sin(\mu t))$, $|\mu| = \alpha$, $|\eta| \le 1$. Taking into account (21), (22), and Lemma 4.1, we obtain

$$\begin{aligned} I(C, g, k) &= \lim_{T\to+\infty} \frac{1}{T} \int_T^{2T} \frac{g(t)}{e^{rt}t^{k-1}}(C_a(t) + C_b(t))\,dt = \\ &= \frac{1}{(k-1)!}(0; 0; \dots ; 0; MC_{X_1}; 0; 0; \dots ; 0)' \in \mathrm{LS}(\Gamma), \end{aligned} \tag{23}$$

where

$$M = (\eta/2)\,\mathrm{Id} \quad \text{for} \quad g(t) = 1 + \eta\cos(\pm\alpha t), \tag{24}$$
$$M = \pm(\eta/2)\mathrm{J} \quad \text{for} \quad g(t) = 1 + \eta\sin(\pm\alpha t). \tag{25}$$

By virtue of the fact that the convex conic hull of vectors (23) for the matrices M of the form (24) and (25), $|\eta| \le 1$, is the plane $\operatorname{span}(x_k, y_k)$, we obtain $\operatorname{span}(x_k, y_k) \subset \mathrm{LS}(\Gamma)$.

(B) We take $v(t) \in \mathrm{LS}(\Gamma)$ equal to

$$\left(0, \dots, 0, \sigma_\alpha(t) \left(\frac{t^{k-1}}{(k-1)!} C_{X_1} + \frac{t^{k-2}}{(k-2)!} C_{X_2} + \ldots + C_{X_k} \right), 0, \dots, 0 \right)',$$

i.e., to the component of vector (22) in the plane $\operatorname{span}(x_k, y_k)$, and repeat the limit passage described in (A) replacing $I(C, g, k)$ with $I(C, g, k-1, v)$, and obtain $\operatorname{span}(x_{k-1}, y_{k-1}) \subset \mathrm{LS}(\Gamma)$.

(C) We repeat process (B) with $I(C, g, p, v)$, where $v(t)$ is the component of vector (22) in the plane $\operatorname{span}(x_p, y_p)$, and p is decreasing from $k-2$ until $l+1$ and obtain the inclusion $\operatorname{span}(x_{l+1}, y_{l+1}; \dots; x_k, y_k) \subset \mathrm{LS}(\Gamma)$.

(D) We apply process (C) with $p = l$ and using the functions $g(t)$ of the form $1 + \eta\cos(\lambda t)$, $1 + \eta\sin(\lambda t)$ for $|\lambda| = \alpha$ and $|\lambda| = \beta$ and obtain $\operatorname{span}(x_l, y_l; z_l, w_l) \subset \mathrm{LS}(\Gamma)$.

(E) We decrease p and repeat procedure (D) until $p = 1$ to obtain

$$\begin{aligned} &L^{(1)}(a) \oplus L^{(1)}(b) = \\ &\quad = \operatorname{span}(x_1, y_1; z_1, w_1; \dots; x_l, y_l; z_l, w_l; x_{l+1}, y_{l+1}; \dots; x_k, y_k) \subset \mathrm{LS}(\Gamma). \end{aligned}$$

In view of (16), the proof of the lemma is completed. $\square$

We can give several sufficient conditions for an element B not to have real N-pairs of eigenvalues. These conditions can be verified simply by the picture of spectrum of the operator $\operatorname{ad} B|_{L^{(1)}}$ in the complex plane. We need them to obtain Corollary 2.

Lemma 4.3. *Suppose that $B \notin L^{(1)}$ and $L_r^{(1)} = L_r^{(2)}$. Then any one of the following conditions is sufficient for the operator $\operatorname{ad} B|_{L^{(1)}}$ not to have real N-pairs of eigenvalues*:

(1) $\mathrm{Sp}_r^{(1)} = \emptyset$,
(2) $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$,
(3) $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

Proof. The first case ($\mathrm{Sp}_r^{(1)} = \emptyset$) is obvious as there are no real eigenvalues at all.

Case 2. Let (α, β) be a real N-pair, $0 < \alpha \le \beta$. We have

$$L^{(2)}(\alpha) \subset L^{(2)} = [L^{(1)}, L^{(1)}] = \sum \{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)} \}.$$

Jacobi's identity implies that $[L^{(1)}(a), L^{(1)}(b)] \subset L^{(1)}(a+b)$, and the spaces $L^{(1)}(c)$, $c \in \mathrm{Sp}^{(1)}$, form a direct sum, that is why

$$L^{(2)}(\alpha) \subset \sum \{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)},\ a + b = \alpha \}. \qquad (26)$$

From the conditions $a+b=\alpha$ and $\operatorname{Re} a>0$, $\operatorname{Re} b>0$ we obtain $\operatorname{Re} a<\alpha$, $\operatorname{Re} b<\alpha$. That is why (26) gives

$$L^{(2)}(\alpha)\subset\sum\{\,[L^{(1)}(a),L^{(1)}(b)]\mid a,b\in\operatorname{Sp}^{(1)},\ \operatorname{Re} a,\operatorname{Re} b<\alpha\,\}.$$

This contradicts item (2) of Definition 3.

Case 3 is considered analogously. □

§ 4.4. Proofs of the sufficient controllability conditions.

Proof of Theorem 2. We show that $\operatorname{LS}(\Gamma)\supset L^{(1)}$.

Introduce the following numbers and sets:

$$n=\min\{\,\operatorname{Re} a\mid a\in\operatorname{Sp}^{(1)},\ L^{(1)}(a)\not\subset\operatorname{LS}(\Gamma)\,\},\tag{27}$$

$$m=\max\{\,\operatorname{Re} a\mid a\in\operatorname{Sp}^{(1)},\ L^{(1)}(a)\not\subset\operatorname{LS}(\Gamma)\,\},\tag{28}$$

$$N=\{\,a\in\operatorname{Sp}^{(1)}\mid \operatorname{Re} a=n,\ L^{(1)}(a)\not\subset\operatorname{LS}(\Gamma)\,\},$$

$$M=\{\,a\in\operatorname{Sp}^{(1)}\mid \operatorname{Re} a=m,\ L^{(1)}(a)\not\subset\operatorname{LS}(\Gamma)\,\}.$$

Suppose that $\operatorname{LS}(\Gamma)\not\supset L^{(1)}$, then $-\infty<n\le m<+\infty$ and $N\ne\emptyset$, $M\ne\emptyset$.

Recall that we have the following decomposition of the vector A corresponding to root subspaces of the operator $\operatorname{ad} B|_{L^{(1)}}$:

$$A=A_B+\sum\{\,A(a)\mid a\in\operatorname{Sp}^{(1)},\ \operatorname{Im} a\ge 0\,\}.$$

Define the element

$$A_1=A-A_B-\sum\{\,A(a)\mid a\in\operatorname{Sp}^{(1)},\ \operatorname{Re} a>m\text{ or }\operatorname{Re} a<n,\ \operatorname{Im} a\ge 0\,\}.$$

Notice that $A_1\in\operatorname{LS}(\Gamma)$ since all terms in the right-hand side belong to $\operatorname{LS}(\Gamma)$. In addition, we have $A_1\in L^{(1)}$. Consider the decomposition

$$A_1=\sum\{\,A_1(a)\mid a\in\operatorname{Sp}^{(1)},\ \operatorname{Im} a\ge 0\,\}.$$

For any $a\in\operatorname{Sp}^{(1)}$ we have

$$\operatorname{Re} a\in[n;m]\Rightarrow A_1(a)=A(a),$$
$$\operatorname{Re} a\notin[n;m]\Rightarrow A_1(a)=0.$$

According to condition 6 of this theorem, the pair of real numbers (n,m) is not an N-pair. That is why at least one of conditions (1)–(3) of Definition 3 is violated. Now we consider these cases separately and come to a contradiction.

(1) Let condition (1) of Definition 3 be violated, i.e., $n\notin\operatorname{Sp}^{(1)}$ or $m\notin\operatorname{Sp}^{(1)}$. Suppose, for definiteness, that $m\notin\operatorname{Sp}^{(1)}$.

Apply Lemma 4.2 with $C = A_1$ and $r = m$. Then we have

$$\begin{aligned}\mathrm{LS}(\Gamma) \supset \sum\{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, a = m,\ a \neq m \,\} = \\ = \sum\{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, a = m \,\},\end{aligned}$$

which is a contradiction to (28).

If $n \notin \mathrm{Sp}^{(1)}$, we come to a contradiction with (27) analogously. That is why case (1) is impossible.

(2) Let now $n, m \in \mathrm{Sp}^{(1)}$ and let condition (2) of Definition 3 be violated, i.e.,

$$L^{(2)}(n) \subset \sum\{\, [L^{(1)}(\lambda), L^{(1)}(\mu)] \mid \lambda, \mu \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\,\lambda, \mathrm{Re}\,\mu \notin [n; m] \,\}.$$

But for $\mathrm{Re}\,\lambda, \mathrm{Re}\,\mu \notin [n; m]$ we have $L^{(1)}(\lambda), L^{(1)}(\mu) \subset \mathrm{LS}(\Gamma)$ (by definitions (27) and (28)), consequently, $L^{(2)}(n) \subset \mathrm{LS}(\Gamma)$. According to hypotheses of this theorem, $L^{(1)}(n) = L^{(2)}(n)$, that is why

$$L^{(1)}(n) \subset \mathrm{LS}(\Gamma). \tag{29}$$

Consider the vector $A_2 = A_1 - A_1(n)$. We have $A_2 \in \mathrm{LS}(\Gamma) \cap L^{(1)}$ and $A_2(n) = 0$. Now we apply Lemma 4.2 with $C = A_2$ and $r = n$ and obtain

$$\mathrm{LS}(\Gamma) \supset \sum\{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, a = n,\ a \neq n \,\}.$$

Then, by virtue of (29), we have

$$\mathrm{LS}(\Gamma) \supset \sum\{\, L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \mathrm{Re}\, a = n \,\},$$

which is a contradiction with (27). That is why case (2) is impossible, and condition (2) of Definition 3 cannot be violated.

(3) We prove analogously that condition (3) of Definition 3 cannot be violated as well.

Hence, all three conditions of Definition 3 hold, and (n, m) is a real N-pair of eigenvalues. This is a contradiction with condition (6) of this theorem. That is why $\mathrm{LS}(\Gamma) \supset L^{(1)}$. But $L = \mathbb{R}B \oplus L^{(1)}$ and $\mathbb{R}B \subset \mathrm{LS}(\Gamma)$. So $\mathrm{LS}(\Gamma) = L$, and Γ is controllable by the controllability condition (15). □

Proof of Corollary 2 follows immediately from Theorem 2 and Lemma 4.3. □

Proof of Corollary 3 is obvious in view of Corollary 2. □

5. Examples and applications

§ 5.1. Metabelian groups. Solvable Lie algebras L having the derived series of length 2:

$$L \supset L^{(1)} \supset L^{(2)} = \{0\},$$

are called *metabelian*. A Lie group with a metabelian Lie algebra is also called *metabelian*.

Our previous results make it possible to obtain controllability conditions for metabelian Lie groups.

Theorem 3. *Let G be a metabelian Lie group. Then the following conditions are sufficient for controllability of a system Γ on G:*

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $\mathrm{Sp}_r^{(1)} = \emptyset$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(5) $\mathrm{top}\,(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$.

If the group G is simply connected, then conditions (1)–(5) *are also necessary for controllability of the system Γ on G.*

The notation $\mathrm{top}\,(A, a)$ used in Theorem 3 is explained in Definition 2 of Sec. 2.

Proof. The sufficiency follows from Corollary 2.

In order to prove the necessity for the simply connected G suppose that Γ is controllable.

Conditions (1) and (2) follow then from items (1) and (2) of Theorem 1.

Condition (3) follows from item (3) of Theorem 1 and from the metabelian property of G:

$$L_r^{(1)} = L_r^{(2)} \subset L^{(2)} = \{0\}.$$

Condition (4). For any $a \in \mathrm{Sp}_c^{(1)}$ we have $L^{(2)}(a) = \{0\}$, that is why $\mathrm{j}\,(a)$ is equal to geometric multiplicity of the eigenvalue a of the operator $\mathrm{ad}\, B|_{L^{(1)}(a)}$, i.e., to $\dim L_c(a)$. By item (6) of Theorem 1, we have $\mathrm{j}\,(a) = 1$, that is why $\dim L_c(a) = 1$.

Condition (5). For any $a \in \mathrm{Sp}_c^{(1)}$ we have $\mathrm{j}\,(a) = 1$, then, by item (7) of Theorem 1, we obtain $\mathrm{top}\,(A, a) \neq 0$. $\square$

Example. Let l be a finite-dimensional real Lie algebra acting linearly in a finite-dimensional real vector space V. Consider their semidirect product $L = V \oplus_{\mathrm{s}} l$. It is a subalgebra of the Lie algebra of affine transformations of the space V since $L \subset V \oplus_{\mathrm{s}} \mathrm{gl}\,(V)$. If l is Abelian, then L is metabelian:

$$L^{(1)} = lV \oplus_{\mathrm{s}} \{0\}, \quad L^{(2)} = \{0\}.$$

In the following subsection we study in detail a particular case when l is one-dimensional.

§ 5.2. Matrix group. Now we apply the controllability conditions from the previous subsection to some particular metabelian matrix group. To begin with we describe this group.

Let V be a real finite-dimensional vector space, $\dim V = n$, and M a linear operator in V. The required metabelian Lie algebra is the semi-direct product

$$L(M) = V \oplus_{\mathrm{s}} \mathbb{R}M$$

(compare with the example at the end of the previous subsection).

Now we choose and fix a base in V, and denote the matrix of the operator M in this base by the same letter M. Then $L(M)$ can be represented as the subalgebra of $\mathrm{gl}\,(n+1,\mathbb{R})$ generated by the following matrices:

$$x = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \quad y_i = E_{i,n+1},\ i = 1, \dots, n.$$

(Recall that E_{ij} is the $n \times n$ matrix with the only unit entry in the ith line and the jth raw.) Obviously, we have

$$L = \mathrm{span}(x; y_1, \dots, y_n), \quad \dim L = n+1,$$
$$L^{(1)} = \mathrm{span}(y_1, \dots, y_n), \quad \dim L^{(1)} = n,$$
$$L^{(2)} = \{0\}.$$

Notice also that $[y_i, y_j] = 0$ for all $i, j = 1, \dots, n$ and M is the matrix of the adjoint operator $\mathrm{ad}\, x|_{L^{(1)}}$ in the base $\{ y_1, \dots, y_n \}$. In the sequel we consider the Lie algebra $L(M) \subset \mathrm{gl}\,(n+1,\mathbb{R})$ in this matrix representation.

Let $G(M)$ be the connected Lie subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ corresponding to $L(M)$. The group $G(M)$ can be parametrized by the matrices

$$g(t,s) = \begin{pmatrix} \exp(Mt) & s \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\ s \in \mathbb{R}^n.$$

It is a semidirect product:

$$G = \mathbb{R}^n \otimes_{\mathrm{s}} G_1, \quad G_1 = \{ \exp(Mt) \mid t \in \mathbb{R} \}.$$

The group $G(M)$ is not simply connected iff the one-parameter subgroup G_1 is periodic, which occurs iff the matrix M has purely imaginary commensurable spectrum. More precisely, we say that a set of numbers $(b_1, \dots, b_n) \in \mathbb{R}^n$ is *commensurable* if

$$(b_1, \dots, b_n) = r \cdot (k_1, \dots, k_n) \quad \text{for some } r \in \mathbb{R},\ (k_1, \dots, k_n) \in \mathbb{Z}^n.$$

And the group $G(M)$ is not simply connected iff

$$\left.\begin{array}{l}\mathrm{Sp}(M) \subset \sqrt{-1}\cdot\mathbb{R}, \\ \text{the set } \mathrm{Im}\,(\mathrm{Sp}(M)) \text{ is commensurable.}\end{array}\right\} \tag{30}$$

Before studying controllability conditions for the group $G(M)$ we present an auxiliary proposition, which translates the Kalman condition (equivalent both to controllability and to rank controllability condition for linear systems $\dot{x} = Ax + ub$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$) into the language of eigenvalues of the matrix A and of components of the vector b in the corresponding root spaces). We will apply this proposition below to reformulate our controllability conditions for right-invariant and bilinear systems.

Lemma 5.1. *Let A be a real $n \times n$ matrix, $b \in \mathbb{R}^n$. Then the Kalman condition*

$$\mathrm{rank}\,(b, Ab, \dots, A^{n-1}b) = n \tag{31}$$

is equivalent to the following conditions:

(1) *the matrix A has a geometrically simple spectrum,*
(2) $\mathrm{top}\,(b,\lambda) \neq 0$ *for any eigenvalue $\lambda \in \mathrm{Sp}(A)$.*

By analogy with Definition 2 in Sec. 2, we say that $\mathrm{top}\,(b,\lambda) \neq 0$ if the component $b(\lambda)$ of the vector b in the root space $\mathbb{R}^n(\lambda)$ corresponding to the eigengalue λ satisfies the condition

$$b(\lambda) \notin (A - \lambda\,\mathrm{Id})\mathbb{R}^n(\lambda),$$

i.e., the vector $b(\lambda)$ has a nonzero component corresponding to the highest adjoined vector in the (single) Jordan chain of the operator A corresponding to λ.

To prove Lemma 5.1, we cite the following

Proposition 3. (Hautus Lemma, [22], Lemma 3.3.7.) *Let A be a complex $n \times n$ matrix, $b \in \mathbb{C}^n$. Then the Kalman condition* (31) *is equivalent to the condition*

$$\mathrm{rank}\,(\lambda \cdot \mathrm{Id} - A, b) = n \quad \forall\ \lambda \in \mathrm{Sp}(A). \tag{32}$$

Proof of Lemma 5.1. In view of Proposition 3, we prove that condition (32) is equivalent to conditions (1), (2) of Lemma 5.1.

First, we suppose that all eigenvalues of A are real; otherwise we pass to complexification. Second, the Kalman condition (31) preserves under changes of base in $\mathbb{R}^n$. That is why we assume that the matrix A is in the Jordan normal form:

$$A = \begin{pmatrix} J_{i_1}(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{i_k}(\lambda_k) \end{pmatrix}, \quad \{\lambda_1, \dots, \lambda_k\} = \operatorname{Sp}(A),$$

$$J_{i_l}(\lambda_l) = \underbrace{\begin{pmatrix} \lambda_l & 0 & \cdots & 0 & 0 \\ 1 & \lambda_l & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda_l & 0 \\ 0 & 0 & \cdots & 1 & \lambda_l \end{pmatrix}}_{i_l}, \quad l = 1, \dots, k.$$

Then the $n \times (n+1)$ matrix in condition (32) is represented as

$$\begin{aligned} \phi(\lambda) &:= (\lambda \cdot \operatorname{Id}_n - A, b) = \\ &= \begin{pmatrix} \lambda \cdot \operatorname{Id}_{i_1} - J_{i_1}(\lambda_1) & \cdots & 0 & b(\lambda_1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda \cdot \operatorname{Id}_{i_k} - J_{i_k}(\lambda_k) & b(\lambda_k) \end{pmatrix}, \end{aligned} \tag{33}$$

where $b(\lambda_l)$, $l = 1, \dots, k$, denotes projection of the vector b onto the root space of the matrix A corresponding to the eigenvalue λ_l.

Necessity. We assume that $\operatorname{rank} \phi(\lambda) = n$ for all $\lambda \in \operatorname{Sp}(A)$ and prove conditions (1), (2) of Lemma 5.1.

1. If spectrum of A is not geometrically simple, then $\lambda_i = \lambda_j$ for some $i \neq j$. Then the matrix $\phi(\lambda_i)$ has two zero columns, and $\operatorname{rank} \phi(\lambda_i) < n$.

2. Suppose that the vector b has the zero λ-top for some $\lambda \in \operatorname{Sp}(A)$; for definiteness, let $\operatorname{top}(b, \lambda_1) = 0$. Then the first component of b in the chosen Jordan base equals to zero, and the first raw of the matrix $\phi(\lambda_1)$ is zero. Hence $\operatorname{rank} \phi(\lambda_1) < n$.

Sufficiency. If conditions (1), (2) of Lemma 5.1 hold, then it is easy to see from representation (33) that all matrices $\phi(\lambda_l)$, $l = 1, \dots, k$, have n linearly independent columns and condition (32) is satisfied. □

Now we obtain controllability conditions for the universal covering $\widetilde{G(M)}$ and for the group $G(M)$ itself.

Theorem 4. *Let M be an $n \times n$ matrix, $G = \widetilde{G(M)}$, $L = L(M)$. A system $\Gamma = A + \mathbb{R}B \subset L$ is controllable on G if and only if the following conditions hold*:

(1) *the matrix M has a purely complex geometrically simple spectrum,*
(2) *$B \notin L^{(1)}$,*

(3) $\operatorname{top}(A, \lambda) \neq 0$ *for all* $\lambda \in \operatorname{Sp}(M)$.

For the group $G(M)$ *conditions* (1)–(3) *are sufficient for controllability; if conditions* (30) *are violated, then* (1)–(3) *are equivalent to controllability on* $G(M)$.

The notation $\operatorname{top}(A, \lambda)$ used in Theorem 4 is explained in Definition 2 in Sec. 2.

Remark. By Lemma 5.1, conditions (1)–(3) of the above theorem are equivalent to the following ones:

(1) the matrix M has a purely complex spectrum,
(2) $B \notin L^{(1)}$,
(3) $\operatorname{span}(B, (\operatorname{ad} B)A, \dots, (\operatorname{ad} B)^{n-1}A) = L$.

Proof of Theorem 4. Theorem 3 (see Subsec. 5.1) is applicable to the group $G = \widetilde{G(M)}$, and condition (1) of Theorem 3 is satisfied.

Decompose the vector $B \in L$ using the base of L:

$$B = B_x x + B_{y_1} y_1 + \ldots + B_{y_n} y_n.$$

$B \notin L^{(1)}$ is equivalent to $B_x \neq 0$. Moreover, in view of the metabelian property of L,

$$\operatorname{Sp}^{(1)} = \operatorname{Sp}(\operatorname{ad} B|_{L^{(1)}}) = B_x \cdot \operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = B_x \cdot \operatorname{Sp}(M).$$

By virtue of Theorem 3, the system Γ is controllable on G if and only if the following conditions hold:

(1) $B \notin L^{(1)}$,
(2) $\operatorname{Sp}(M) \cap \mathbb{R} = \emptyset$,
(3) the matrix M has a geometrically simple spectrum,
(4) $\operatorname{top}(A, \lambda) \neq 0$ for all $\lambda \in \operatorname{Sp}(M)$.

Now the proposition of the current theorem for $\widetilde{G(M)}$ follows.

For $G(M)$, controllability is implied by controllability on its universal covering $\widetilde{G(M)}$. And if conditions (30) are violated, then $G(M) = \widetilde{G(M)}$. $\square$

Let now conditions (30) be satisfied. Then the group $G(M)$ is a semi-direct product of the vector group $\mathbb{R}^n$ and the one-dimensional compact group G_1. But controllability conditions on such semi-direct products were obtained by B. Bonnard, V. Jurdjevic, I. Kupka, and G. Sallet [6]: if the compact group has no fixed nonzero points in the vector group (which is just the case), then the controllability is equivalent to the rank controllability condition (Theorem 1, [6]).

So we have complete controllability conditions of systems of the form $\Gamma = A + \mathbb{R}B$ on the group $G(M)$ and its simply connected covering $\widetilde{G(M)}$. In the simply connected case (i.e., when conditions (30) are violated) we have Theorem 4, and otherwise the theorem of B. Bonnard, V. Jurdjevic, I. Kupka, and G. Sallet [6] works.

§ 5.3. Bilinear system. Now we apply the controllability conditions for the group $G(M)$ and study global controllability of the bilinear system

$$\dot{X} = uAX + b, \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{Σ}$$

where A is a constant real $n \times n$ matrix and $b \in \mathbb{R}^n$.

Theorem 5. *The system Σ is globally controllable on $\mathbb{R}^n$ if and only if the following conditions hold*:

(1) *the matrix A has a purely complex spectrum,*
(2) $\operatorname{span}(b, Ab, \dots, A^{n-1}b) = \mathbb{R}^n$.

Remark. By Lemma 5.1, conditions (1)–(2) of this theorem can equivalently be formulated as follows:

(1) the matrix A has a purely complex geometrically simple spectrum,
(2) $\operatorname{top}(b, \lambda) \neq 0$ for all $\lambda \in \operatorname{Sp}(A)$.

Proof of Theorem 5. We use the hypotheses of this theorem in the equivalent form given in the above remark.

Sufficiency. Consider the bilinear system

$$\dot{Y} = \overline{A}Y + u\overline{B}Y, \quad Y = (X, 1)' \in \mathbb{R}^{n+1}, \quad u \in \mathbb{R}, \tag{$\overline{\Sigma}$}$$

where

$$\overline{A} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

are $(n + 1) \times (n + 1)$ matrices. It is easy to see that the system Σ is globally controllable on $\mathbb{R}^n$ iff the system $\overline{\Sigma}$ is globally controllable in the n-dimensional affine plane

$$(\mathbb{R}^n, 1)' = \{\, Y = (X, 1)' \in \mathbb{R}^{n+1} \mid X \in \mathbb{R}^n \,\}.$$

Consider the matrix Lie algebra $L(A)$ and the corresponding Lie group $G(A)$ described in the previous subsection. We have $\overline{A}, \overline{B} \in L(A)$, and $\overline{\Gamma} = \overline{A} + \mathbb{R}\overline{B} \subset L(A)$ is a right-invariant system on the group $G(A)$. Theorem 4 ensures that under hypotheses (1), (2) of the current theorem the system $\overline{\Gamma}$ is controllable on the group $G(A)$. But the group $G(A)$ acts transitively in the plane $(\mathbb{R}^n, 1)'$, and the bilinear system $\overline{\Sigma}$ is the projection of the right-invariant system $\overline{\Gamma}$ from the group $G(A)$ onto the plane $(\mathbb{R}^n, 1)'$. That is why controllability of $\overline{\Gamma}$ on $G(A)$ implies controllability of $\overline{\Sigma}$ on $(\mathbb{R}^n, 1)'$. Thus Σ is globally controllable on $\mathbb{R}^n$.

Necessity. Assume that Σ is globally controllable on $\mathbb{R}^n$.

(1a) First we show that the matrix A has no real eigenvalues. Suppose there is at least one eigenvalue $a \in \mathrm{Sp}(A) \cap \mathbb{R}$. We choose a Jordan base $\{ e_1, \dots, e_n \}$ of the matrix A and denote by $\{ x_1, \dots, x_n \}$ the corresponding coordinates in $\mathbb{R}^n$. Let e_k denote the maximum order root vector coresponding to the eigenvalue a:

$$(A - a\,\mathrm{Id})^k e_k = a e_k + \varepsilon e_{k+1}, \quad \varepsilon = 1 \text{ or } 0,$$

and k is the maximal possible integer. Then the system Σ implies

$$\dot{x}_k = u a x_k + b_k,$$

where b_k is the kth coordinate of the vector b in the base $\{ e_1, \dots, e_n \}$. Now it is obvious that at least one of the half-spaces $\{ x_k \geq 0 \}$, $\{ x_k \leq 0 \}$ is positive invariant for the system Σ, i.e., this system is not controllable.

(1b) Now we show that the spectrum $\mathrm{Sp}(A)$ is geometrically simple. Suppose that for some (complex) eigenvalue $\lambda \in \mathrm{Sp}(A)$ there are at least two linearly independent eigenvectors. Then we apply the same transformation of Jordan chains as in Lemma 3.4 to obtain the zero component of the vector b in the two-dimensional subspace of $\mathbb{R}^n$ spanned by the pair of the highest order root vectors of the matrix A (see conditions (13), (14)). Now if x_k, y_k are the coordinates in $\mathbb{R}^n$ in the transformed Jordan base corresponding to the above-mentioned two-dimensional subspace, then the system Σ yields

$$\begin{aligned} \dot{x}_k &= u(\alpha x_k + \beta y_k), \\ \dot{y}_k &= u(-\beta x_k + \alpha y_k), \end{aligned}$$

where $\alpha = \mathrm{Re}\,\lambda$, $\beta = \mathrm{Im}\,\lambda$. Hence it follows that the codimension two subspace $\{ x_k = y_k = 0 \}$ is (both positive and negative) invariant for the system Σ, and so it is not controllable.

(2) Finally, we show that the vector b has a nonzero λ-top for any eigenvalue $\lambda \in \mathrm{Sp}(A)$. If this is not the case, we choose any Jordan chain in the root space corresponding to λ, apply the argument from item 1.b) above, and show that Σ is not controllable.

The necessity and sufficiency are now completely proved. $\square$

§ 5.4. The Euclidean group in two dimensions. It is interesting to consider the work of the above general theory for the visual three-dimensional case.

Let $G = G(\mathrm{J}\,) = \mathrm{E}(2)$ be the Euclidean group of motions of the plane $\mathbb{R}^2$. $\mathrm{E}(2)$ is connected but not simply connected. It can be represented as the group of 3×3 matrices of the form

$$g(t, s_1, s_2) = \begin{pmatrix} \cos t & -\sin t & s_1 \\ \sin t & \cos t & s_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2) \quad \text{for} \quad t \in \mathbb{R}, \qquad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathbb{R}^2.$$

The corresponding matrix Lie algebra L is spanned by the matrices

$$x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the system $\Gamma = A + \mathbb{R}B$ on $\widetilde{\mathrm{E}(2)}$ — the universal covering of $\mathrm{E}(2)$. A complete characterization of controllability of Γ on $\widetilde{\mathrm{E}(2)}$ is derived from Theorem 4.

Theorem 6. *The system Γ is controllable on $\widetilde{\mathrm{E}(2)}$ if and only if the vectors A, B are linearly independent and $B \notin \mathrm{span}(y, z)$.*

Let us compare the controllability conditions for $\widetilde{\mathrm{E}(2)}$ with the following conditions for $\mathrm{E}(2)$ derived from Theorem 1, [6]:

Theorem 7. *The system Γ is controllable on $\mathrm{E}(2)$ if and only if the vectors A, B are linearly independent and $\mathrm{span}(A, B) \not\subset \mathrm{span}(y, z)$.*

Finally, Theorem 5 gives the following geometrically clear proposition.

Theorem 8. *The system*

$$\dot{X} = uAX + b, \quad X, b \in \mathbb{R}^2, \quad u \in \mathbb{R}$$

is controllable on the plane $\mathbb{R}^2$ if and only if:

(1) *the matrix A has a purely complex spectrum,*
(2) $b \neq 0$.

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