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Extremal trajectories in a nilpotent sub-Riemannian problem on the Engel group

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Abstract. On the Engel group a nilpotent sub-Riemannian problem is considered, a 4-dimensional optimal control problem with a 2-dimensional linear control and an integral cost functional. It arises as a nilpotent approximation to nonholonomic systems with 2-dimensional control in a 4-dimensional space (for example, a system describing the navigation of a mobile robot with trailer). A parametrization of extremal trajectories by Jacobi functions is obtained. A discrete symmetry group and its fixed points, which are Maxwell points, are described. An estimate for the cut time (the time of the loss of optimality) on extremal trajectories is derived on this basis.

Bibliography: 25 titles.

Keywords: optimal control, sub-Riemannian geometry, geometric methods, Engel group.

§ 1. Introduction

This paper is concerned with the analysis of a nilpotent sub-Riemannian problem on the Engel group, a 4-dimensional optimal control problem with a 2-dimensional linear control and an integral performance functional. Nilpotent sub-Riemannian problems are fundamental for sub-Riemannian geometry since they provide a local quasi-homogeneous approximation to general sub-Riemannian problems (see [1]–[4]). For instance, a nilpotent sub-Riemannian problem on the 3-dimensional Heisenberg group (see [5]) is a cornerstone of the entire sub-Riemannian geometry. Invariant sub-Riemannian problems on Lie groups have intensively been investigated through the last 10 years by means of geometric control theory (see [6]–[12]).

The invariant sub-Riemannian problem on the Engel group has several important properties which underline its special role in sub-Riemannian geometry. First, this is the simplest sub-Riemannian problem with nontrivial abnormal extremal trajectories (it is known that in 3-dimensional contact problems abnormal extremal trajectories are constant [13]). Second, this problem projects onto the sub-Riemannian problem in the Martinet flat case [14], so the problem on the Engel group is the simplest invariant sub-Riemannian problem on a nilpotent Lie group with nonsubanalytic sub-Riemannian sphere. Third, the vector distribution in this problem

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is not 2-generating [15]: the growth vector $(2,3,4)$ of this problem has length 3. So, in particular, it is the simplest sub-Riemannian problem lacking the property of interlacing of conjugate points and Maxwell points, which is characteristic for sub-Riemannian problems with 2-generating distributions.

The invariant problem on an Engel group is important for applications, for instance, in robotics since it gives a nilpotent approximation for the system describing the navigation of a mobile robot with trailer in the plane or on a 2-dimensional surface (see [2] and [16]).

For all these reasons the importance of investigations of the invariant sub-Riemannian problem on the Engel group is beyond all doubt. In our paper we use in this problem new methods of geometric control theory, which have proved to be successful in the recent work concerned with the Euler elastic problem [17], [18], the nilpotent sub-Riemannian problem with growth vector $(2,3,5)$ [9], a sub-Riemannian problem on the group of plane motions [12] and the problem of rolling a sphere over the plane [10].

The structure of this paper is as follows. In § 2 we set the problem and discuss its statement. In § 3 we apply Pontryagin's maximum principle to the problem, and in § 4 and § 5 find a parametrization of extremal trajectories; in particular, in § 5 we describe the exponential map providing a parametrization for all the extremal trajectories. In § 6 we describe discrete symmetries of the exponential map and in § 7 investigate the corresponding Maxwell points, which are fixed points of these symmetries. On this basis, in Theorem 3 we establish the central result of the paper, an upper estimate for the cut time on extremal trajectories (when the trajectory loses the property of being optimal).

§ 2. Setting the optimal control problem

We look at the following optimal control problem:

$$\dot q = \begin{pmatrix} \dot x \\ \dot y \\ \dot z \\ \dot v \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\dfrac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \dfrac{x}{2} \\ \dfrac{x^2+y^2}{2} \end{pmatrix}, \qquad q \in \mathbb{R}^4, \quad u \in \mathbb{R}^2, \tag{2.1}$$

with the boundary conditions

$$q(0) = q_0 = (x_0, y_0, z_0, v_0), \qquad q(t_1) = q_1 = (x_1, y_1, z_1, v_1) \tag{2.2}$$

and performance functional

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min, \tag{2.3}$$

where the point $q = (x, y, z, v) \in \mathbb{R}^4 = M$ determines the state of the system, $u = (u_1, u_2)$ is the control, and the terminal time t_1 is fixed.

We introduce our notation for the vector fields multiplying the controls on the right-hand side of (2.1):

$$X_1 = \left(1, 0, -\frac{y}{2}, 0\right)^T, \qquad X_2 = \left(0, 1, \frac{x}{2}, \frac{x^2+y^2}{2}\right)^T,$$

and find the commutators of these fields:

$$X_3 = [X_1, X_2] = \frac{\partial X_2}{\partial q} X_1 - \frac{\partial X_1}{\partial q} X_2 = (0, 0, 1, x)^T,$$
$$X_4 = [X_1, X_3] = \frac{\partial X_3}{\partial q} X_1 - \frac{\partial X_1}{\partial q} X_3 = (0, 0, 0, 1)^T.$$

At each point $q \in \mathbb{R}^4$ the vector fields $X_1(q), \dots, X_4(q)$ are linearly independent, so by the Rashevskiĭ-Chow theorem (see [15]) the system (2.1) is completely controllable in $\mathbb{R}^4$ (that is, any points $q_0, q_1 \in M$ can be connected by a trajectory of the system).

The fields X_1 and X_2 generate the 4-dimensional nilpotent Lie algebra

$$\operatorname{Lie}(X_1, X_2) = \operatorname{span}(X_1, X_2, X_3, X_4)$$

with the following multiplication table:

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.$$

It is called the *Engel algebra* [1]. We can introduce a group structure (law of multiplication) in $\mathbb{R}^4$, making a Lie group of $\mathbb{R}^4$, so that $X_1, \dots, X_4$ become basic left-invariant fields on this Lie group. It is easy to see that this law of multiplication has the form

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ v_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + \dfrac{x_1 y_2 - x_2 y_1}{2} \\ v_1 + v_2 + \dfrac{y_1 y_2}{2}(y_1 + y_2) + x_1 z_2 + \dfrac{x_1 y_2}{2}(x_1 + x_2) \end{pmatrix}.$$

The space $\mathbb{R}^4$ endowed with this group structure is called the *Engel group*.

Problem (2.1)–(2.3) is a left-invariant sub-Riemannian problem on the Engel group for the sub-Riemannian structure on $\mathbb{R}^4$ defined by the fields X_1 and X_2 as an orthonormal basis. It is known [19] that any two invariant nonholonomic sub-Riemannian problems on the Engel group can be transformed into one another by Lie group homomorphisms of the Engel group, so (2.1)–(2.3) is a concrete model for all the problems in this class.

Since the problem is invariant under left translations of the Engel group, we can assume that the origin is the identity element of the group $q_0 = (x_0, y_0, z_0, v_0) = (0, 0, 0, 0)$.

It is easy to see that (2.1)–(2.3) is equivalent to the following geometric problem. Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ be points in the plane connected by a curve $\gamma_0 \subset \mathbb{R}^2$. Let $S \in \mathbb{R}$ and let $l \subset \mathbb{R}^2$ be a line. Join the points (x_0, y_0) and (x_1, y_1) by a shortest curve $\gamma \subset \mathbb{R}^2$ such that γ_0 and γ together bound a region of area S in the plane such that its centre of mass lies on l.

§ 3. Pontryagin's maximum principle

The existence of optimal trajectories in problem (2.1)–(2.3) is provided by Filippov's theorem (see [15]). It follows from the Cauchy-Schwarz inequality that minimizing the sub-Riemannian length (2.3) is equivalent to minimizing the action

$$\int_0^{t_1} \frac{u_1^2 + u_2^2}{2}\, dt \to \min. \tag{3.1}$$

We obtain the optimal control problem (2.1), (2.2), (3.1), and apply to it Pontryagin's maximum principle (see [20] or [15]). Let us introduce the vector of costate variables $\psi = (\psi_0, \psi_1, \psi_2, \psi_3, \psi_4)$ and the Hamiltonian

$$H(\psi, q, u) = \psi_0 \frac{u_1^2 + u_2^2}{2} + \psi_1 u_1 + \psi_2 u_2 + \psi_3 \frac{x u_2 - y u_1}{2} + \psi_4 \frac{x^2 + y^2}{2} u_2.$$

From Pontryagin's maximum principle for this Hamiltonian we obtain a Hamiltonian system for the costate variables:

$$\dot\psi_1 = -H_x = -\psi_3 \frac{u_2}{2} - \psi_4 x u_2, \qquad \dot\psi_2 = -H_y = \psi_3 \frac{u_1}{2} - \psi_4 y u_2, \qquad \dot\psi_3 = \dot\psi_4 = 0,$$

the maximum condition

$$\max_{u \in \mathbb{R}^2} H\bigl(\psi(t), \hat q(t), u\bigr) = H\bigl(\psi(t), \hat q(t), \hat u(t)\bigr), \qquad \psi_0 \leqslant 0, \tag{3.2}$$

where $\hat u(t)$, $\hat q(t)$ is the optimal process, and the condition

$$\psi(t) \neq 0$$

of the nontriviality of the costate variables.

§ 4. Abnormal extremal trajectories

We shall investigate the abnormal case $\psi_0 = 0$. From the maximum condition (3.2) we obtain

$$H_{u_1} = \psi_1 - \psi_3 \frac{y}{2} = 0, \tag{4.1}$$

$$H_{u_2} = \psi_2 + \psi_3 \frac{x}{2} + \psi_4 \frac{x^2 + y^2}{2} = 0. \tag{4.2}$$

We see from (4.1) that

$$0 = \dot\psi_1 - \psi_3 \frac{u_2}{2} = -u_2(\psi_3 + \psi_4 x),$$

and in a similar way, from (4.2) we obtain

$$0 = \dot\psi_2 + \psi_3 \frac{u_1}{2} + \psi_3(x u_1 + y u_2) = u_1(\psi_3 + \psi_4 x).$$

We can assume that $u_1^2 + u_2^2 = 1$, so that $\psi_3 + \psi_4 x = 0$. If $\psi_4 = 0$, then $\psi_3 = 0$ and therefore $\psi = 0$, in contradiction with the nontriviality of the costate variables. Hence $\psi_4 \neq 0$; this gives us the equations for extremal curves in the abnormal case:

$$x = 0, \qquad y = \pm t, \qquad z = 0, \qquad v = \pm \frac{t^3}{6}. \tag{4.3}$$

The projection of these curves onto the (x, y)-plane gives a straight line.

§ 5. Normal extremal trajectories

5.1. A normal Hamiltonian system. Now we look at the normal case $\psi_0 = -1$. It follows from the maximum condition (3.2) that $H_{u_1} = 0$ and $H_{u_2} = 0$. Hence

$$u_1 = \psi_1 - \psi_3 \frac{y}{2}, \qquad u_2 = \psi_2 + \psi_3 \frac{x}{2} + \psi_4 \frac{x^2 + y^2}{2}.$$

Let $h_i = \langle \psi, X_i \rangle$ be the Hamiltonians corresponding to the basis vector fields X_i in the tangent space $T_q M$ and linear on the fibres of the cotangent space T^*M:

$$h_1 = \psi_1 - \psi_3 \frac{y}{2}, \qquad h_2 = \psi_2 + \psi_3 \frac{x}{2} + \psi_4 \frac{x^2 + y^2}{2}, \qquad h_3 = \psi_3 + \psi_4 x,$$
$$h_4 = \psi_4.$$

Differentiating them while taking account of the Hamiltonian system of the maximum principle we obtain

$$\dot h_1 = -h_2 h_3, \qquad \dot h_2 = h_1 h_3, \qquad \dot h_3 = h_1 h_4, \qquad \dot h_4 = 0.$$

Limiting ourselves to the level surface $\bigl\{H = \frac{1}{2}(h_1^2 + h_2^2) = \frac{1}{2}\bigr\}$ we go over to the coordinate system (θ, c, α) on this surface:

$$h_1 = \cos\biggl(\theta + \frac{\pi}{2}\biggr), \qquad h_2 = \sin\biggl(\theta + \frac{\pi}{2}\biggr), \qquad h_3 = c, \qquad h_4 = \alpha.$$

In the variables $(\theta, c, \alpha, x, y, z, v)$ the Hamiltonian system of the Pontryagin maximum principle assumes the following form in the normal case:

$$\dot\theta = c, \tag{5.1}$$
$$\dot c = -\alpha \sin\theta, \tag{5.2}$$
$$\dot\alpha = 0, \tag{5.3}$$
$$\dot x = -\sin\theta, \tag{5.4}$$
$$\dot y = \cos\theta, \tag{5.5}$$
$$\dot z = \frac{x\cos\theta + y\sin\theta}{2}, \tag{5.6}$$
$$\dot v = \cos\theta\, \frac{x^2 + y^2}{2}. \tag{5.7}$$

Note that the subsystem for the costate variables reduces to the pendulum equation

$$\ddot\theta = -\alpha\sin\theta, \qquad \alpha = \mathrm{const}. \tag{5.8}$$

So the projections of extremal curves onto the (x, y)-plane are Euler elastics, stationary configurations of an elastic rod in the plane (see [17], [18], [21]–[23]).

In the pendulum equation the parameter α has the following physical meaning:

$$\alpha = \frac{g}{L},$$

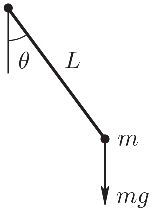

Figure 1. The pendulum (5.8) for $\alpha > 0$.

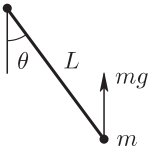

Figure 2. The pendulum (5.8) for $\alpha < 0$.

where g is the gravitational acceleration and L is the length of the pendulum. Thus if $\alpha = 0$, then the pendulum moves in weightlessness; for $\alpha > 0$ the gravity force is pointing downwards (Fig. 1); and for $\alpha < 0$ it is pointing upwards (Fig. 2).

5.2. Partitioning the initial cylinder C. Let us introduce the energy integral of the pendulum (5.8):

$$E = \frac{h_3^2}{2} - h_2 h_4 = \frac{c^2}{2} - \alpha\cos\theta \in [-|\alpha|, +\infty), \qquad \dot{E} = h_3\dot{h}_3 - \dot{h}_2 h_4 = 0.$$

The family of normal extremal trajectories can be parametrized by points in the cylinder

$$\begin{aligned} C &= T^*_{q_0} M \cap \left\{ H = \frac{1}{2} \right\} = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^4 \mid h_1^2 + h_2^2 = 1\} \\ &= \{(\theta, c, \alpha) \mid \theta \in S^1,\ c, \alpha \in \mathbb{R}\}. \end{aligned}$$

We partition C into subsets corresponding to different types of pendulum trajectories:

$$\begin{gathered} C = \bigcup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \varnothing, \quad i \neq j, \quad \lambda = (\theta, c, \alpha), \\ C_1 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (-|\alpha|, |\alpha|)\}, \\ C_2 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (|\alpha|, +\infty)\}, \\ C_3 = \{\lambda \in C \mid \alpha \neq 0,\ E = |\alpha|, c \neq 0\}, \\ C_4 = \{\lambda \in C \mid \alpha \neq 0,\ E = -|\alpha|\}, \\ C_5 = \{\lambda \in C \mid \alpha \neq 0,\ E = |\alpha|,\ c = 0\}, \\ C_6 = \{\lambda \in C \mid \alpha = 0,\ c \neq 0\}, \\ C_7 = \{\lambda \in C \mid \alpha = c = 0\}. \end{gathered}$$

The sets C_i, $i = 1, \dots, 5$, are further subdivided into subsets depending on the sign of α:

$$C_i^+ = C_i \cap \{\alpha > 0\}, \quad C_i^- = C_i \cap \{\alpha < 0\}, \qquad i \in \{1, \dots, 5\}.$$

In addition, C_6, $C_2^\pm$ and $C_3^\pm$ are subdivided into connected components depending on the sign of c:

$$C_{6+} = C_6 \cap \{c > 0\}, \qquad C_{6-} = C_6 \cap \{c < 0\},$$
$$C_{i+}^\pm = C_i^\pm \cap \{c > 0\}, \quad C_{i-}^\pm = C_i^\pm \cap \{c < 0\}, \qquad i \in \{2, 3\}.$$

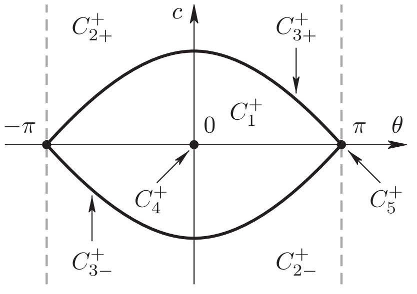

Figure 3. Partitioning C for $\alpha > 0$.

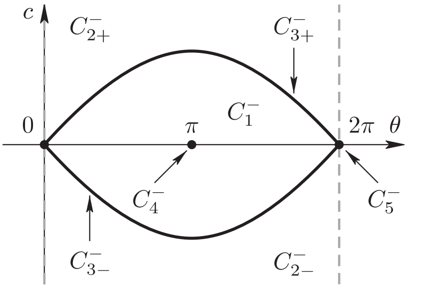

Figure 4. Partitioning C for $\alpha < 0$.

We give the partitioning of a section of the cylinder $\{\lambda \in C \mid \alpha = \text{const} \neq 0\}$ in Fig. 3 (for $\alpha > 0$) and Fig. 4 (for $\alpha < 0$).

5.3. Elliptic coordinate system. To calculate extremal trajectories from the subsets C_1, C_2 and C_3 we introduce coordinates (φ, k, α) in which the subsystem for the costate variables (5.1)–(5.3) is straightened out. Such coordinate systems were used in [9], [17], [10] and [12], in investigations of several related optimal control problems in which the subsystem for the costate variables of Pontryagin's maximum principle reduces to the pendulum equation.

In the domain C_1^+,

$$k = \sqrt{\frac{E+\alpha}{2\alpha}} = \sqrt{\frac{c^2}{4\alpha} + \sin^2\frac{\theta}{2}} \in (0,1),$$
$$\sin\frac{\theta}{2} = k\,\mathrm{sn}(\sqrt{\alpha}\varphi), \qquad \cos\frac{\theta}{2} = \mathrm{dn}(\sqrt{\alpha}\varphi), \qquad \frac{c}{2} = k\sqrt{\alpha}\,\mathrm{cn}(\sqrt{\alpha}\varphi),$$
$$\varphi \in [0, 4K].$$

In the domain C_2^+ we set

$$k = \sqrt{\frac{2\alpha}{E+\alpha}} = \frac{1}{\sqrt{c^2/(4\alpha) + \sin^2(\theta/2)}} \in (0,1),$$
$$\sin\frac{\theta}{2} = \operatorname{sgn} c\,\mathrm{sn}\,\frac{\sqrt{\alpha}\varphi}{k}, \qquad \cos\frac{\theta}{2} = \mathrm{cn}\,\frac{\sqrt{\alpha}\varphi}{k}, \qquad \frac{c}{2} = \operatorname{sgn} c\frac{\sqrt{\alpha}}{k}\,\mathrm{dn}\,\frac{\sqrt{\alpha}\varphi}{k},$$
$$\varphi \in [0, 2kK], \qquad \psi = \frac{\varphi}{k}.$$

On C_3^+,

$$k = 1,$$
$$\sin\frac{\theta}{2} = \operatorname{sgn} c \tanh(\sqrt{\alpha}\varphi), \qquad \cos\frac{\theta}{2} = \frac{1}{\cosh(\sqrt{\alpha}\varphi)}, \qquad \frac{c}{2} = \operatorname{sgn} c \frac{\sqrt{\alpha}}{\cosh(\sqrt{\alpha}\varphi)},$$
$$\varphi \in (-\infty, +\infty).$$

Here and throughout, sn, cn, dn and E are Jacobi elliptic functions (see [24]).

On C_1^-, C_2^- and C_3^- we introduce new coordinate systems as follows:

$$\varphi(\theta, c, \alpha) = \varphi(\theta - \pi, c, -\alpha), \tag{5.9}$$
$$k(\theta, c, \alpha) = k(\theta - \pi, c, -\alpha). \tag{5.10}$$

Immediate differentiation shows that in these coordinates the subsystem for the costate variables (5.1)–(5.3) takes the following form:

$$\dot{\varphi} = 1, \qquad \dot{k} = 0, \qquad \dot{\alpha} = 0,$$

so that it has solutions

$$\varphi(t) = \varphi_t = \varphi + t, \qquad k = \text{const}, \qquad \alpha = \text{const}. \tag{5.11}$$

5.4. Parametrization of extremal trajectories with $\boldsymbol{\lambda \in \bigcup_{i=1}^3 C_i}$ in the case $\boldsymbol{\alpha = 1}$. Let $\alpha = 1$. From the definition of the variables φ and k we obtain the following parametrization of the θ_t-components of extremal trajectories.

If $\lambda \in C_1^+$, then

$$\sin\theta_t = 2k \operatorname{sn}\varphi_t \operatorname{dn}\varphi_t, \qquad \cos\theta_t = 1 - 2k^2 \operatorname{sn}^2\varphi_t.$$

If $\lambda \in C_2^+$, then

$$\sin\theta_t = 2\operatorname{sgn} c \operatorname{sn}\psi_t \operatorname{cn}\psi_t, \quad \cos\theta_t = \operatorname{cn}^2\psi_t - \operatorname{sn}^2\psi_t, \qquad \psi_t = \frac{\varphi + t}{k}.$$

If $\lambda \in C_3^+$, then

$$\sin\theta_t = 2\operatorname{sgn} c \frac{\tanh\varphi_t}{\cosh\varphi_t}, \qquad \cos\theta_t = \frac{1 - \operatorname{sh}^2\varphi_t}{\cosh^2\varphi_t}.$$

Integrating (5.4)–(5.7) we obtain a parametrization for extremal trajectories in the case $\alpha = 1$.

If $\lambda \in C_1$, then

$$x_t = 2k(\operatorname{cn}\varphi_t - \operatorname{cn}\varphi),$$
$$y_t = 2(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) - t,$$
$$z_t = 2k\left(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi - \frac{y_t}{2}(\operatorname{cn}\varphi_t + \operatorname{cn}\varphi)\right),$$
$$v_t = \frac{y_t^3}{6} + 2k^2 \operatorname{cn}^2\varphi y_t - 4k^2 \operatorname{cn}\varphi(\operatorname{sn}\varphi_t \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \operatorname{dn}\varphi) + 2k^2\left(\frac{2}{3}\operatorname{cn}\varphi_t \operatorname{dn}\varphi_t \operatorname{sn}\varphi_t \right.$$
$$\left. - \frac{2}{3}\operatorname{cn}\varphi \operatorname{dn}\varphi \operatorname{sn}\varphi + \frac{1-k^2}{3k^2} t + \frac{2k^2-1}{3k^2}(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi))\right).$$

If $\lambda \in C_2$, then

$$\begin{aligned}
x_t &= \frac{2\operatorname{sgn} c}{k}(\operatorname{dn}\psi_t - \operatorname{dn}\psi),\\
y_t &= \frac{k^2-2}{k^2}t + \frac{2}{k}(\mathrm{E}(\psi_t) - \mathrm{E}(\psi)),\\
z_t &= -\frac{x_t y_t}{2} - \frac{2\operatorname{sgn} c\operatorname{dn}\psi}{k}y_t + 2\operatorname{sgn} c(\operatorname{cn}\psi_t\operatorname{sn}\psi_t - \operatorname{cn}\psi\operatorname{sn}\psi),\\
v_t &= \frac{4}{k}\biggl(\frac{1}{3}\operatorname{cn}\psi_t\operatorname{dn}\psi_t\operatorname{sn}\psi_t - \frac{1}{3}\operatorname{cn}\psi\operatorname{dn}\psi\operatorname{sn}\psi - \frac{1-k^2}{3k^3}t - \frac{k^2-2}{6k^2}(\mathrm{E}(\psi_t) - \mathrm{E}(\psi))\biggr)\\
&\quad + \frac{y_t^3}{6} + \frac{2y_t}{k^2}\operatorname{dn}^2\psi - \frac{4}{k}\operatorname{dn}\psi(\operatorname{cn}\psi_t\operatorname{sn}\psi_t - \operatorname{cn}\psi\operatorname{sn}\psi),\\
\psi &= \frac{\varphi}{k}, \qquad \psi_t = \frac{\psi+t}{k}.
\end{aligned}$$

If $\lambda \in C_3$, then

$$\begin{aligned}
x_t &= 2\operatorname{sgn} c\biggl(\frac{1}{\cosh\varphi_t} - \frac{1}{\cosh\varphi}\biggr),\\
y_t &= 2(\tanh\varphi_t - \tanh\varphi) - t,\\
z_t &= -\frac{x_t y_t}{2} - \frac{2\operatorname{sgn} c}{\cosh\varphi}y_t + 2\operatorname{sgn} c\biggl(\frac{\tanh\varphi_t}{\cosh\varphi_t} - \frac{\tanh\varphi}{\cosh\varphi}\biggr),\\
v_t &= \frac{2}{3}\biggl(\tanh\varphi_t - \tanh\varphi + 2\frac{\tanh\varphi_t}{\cosh^2\varphi_t} - 2\frac{\tanh\varphi}{\cosh^2\varphi}\biggr)\\
&\quad + \frac{y_t^3}{6} + \frac{2y_t}{\cosh^2\varphi} - \frac{4}{\cosh\varphi}\biggl(\frac{\tanh\varphi_t}{\cosh\varphi_t} - \frac{\tanh\varphi}{\cosh\varphi}\biggr).
\end{aligned}$$

5.5. Parametrization of extremal trajectories with $\boldsymbol{\lambda \in \bigcup_{i=1}^{3} C_i}$ in the general case of $\boldsymbol{\alpha \neq 0}$. We obtain a parametrization of extremal trajectories in the general case from the formulae for the special case $\alpha = 1$ using symmetries of the Hamiltonian system (5.1)–(5.7).

5.5.1. *The case* $\alpha > 0$. System (5.1)–(5.7) possesses the symmetry (of dilation type)

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto \biggl(\theta, \frac{c}{\sqrt{\alpha}}, 1, \sqrt{\alpha}x, \sqrt{\alpha}y, \alpha z, \alpha^{3/2}v, \sqrt{\alpha}t\biggr),$$

which transforms the variables φ and k as follows:

$$(\varphi, k, \alpha) \mapsto (\sqrt{\alpha}\varphi, k, 1).$$

Hence extremal trajectories of the case $\alpha > 0$ can be expressed as follows in terms of extremal trajectories of the case $\alpha = 1$ (which we calculated in § 5.4):

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = \biggl(\frac{x_{\sqrt{\alpha}t}}{\sqrt{\alpha}}, \frac{y_{\sqrt{\alpha}t}}{\sqrt{\alpha}}, \frac{z_{\sqrt{\alpha}t}}{\alpha}, \frac{v_{\sqrt{\alpha}t}}{\alpha^{3/2}}\biggr)(\sqrt{\alpha}\varphi, k, 1).$$

5.5.2. *The case* $\alpha < 0$. The Hamiltonian system (5.1)–(5.7) has the symmetry (of reflection type)

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t),$$

which does not change the values of φ and k (see (5.9) and (5.10)). Hence extremal trajectories of the case $\alpha < 0$ can be expressed as follows in terms of extremal trajectories of the case $\alpha > 0$:

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = (-x_t, -y_t, z_t, -v_t)(\varphi, k, -\alpha).$$

5.5.3. *The general case of* $\alpha \neq 0$. We set $\sigma = \sqrt{|\alpha|}$ and $s_1 = \operatorname{sgn} \alpha$. Bearing in mind the results established in the previous subsections we obtain the following expression for extremal trajectories:

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = \left(\frac{s_1}{\sigma} x_{\sigma t}, \frac{s_1}{\sigma} y_{\sigma t}, \frac{1}{\sigma^2} z_{\sigma t}, \frac{s_1}{\sigma^3} v_{\sigma t}\right)(\sigma\varphi, k, 1). \tag{5.12}$$

This gives us parametrizations of extremal trajectories in the general case.

If $\lambda \in C_1$, then

$$\begin{aligned}
x_t &= \frac{2k\sigma}{\alpha}(\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)),\\
y_t &= \frac{2\sigma}{\alpha}(\operatorname{E}(\sigma\varphi_t) - \operatorname{E}(\sigma\varphi)) - \operatorname{sgn}\alpha t,\\
z_t &= \frac{2k}{|\alpha|}\left(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi) - \frac{\sigma k y_t}{2\alpha}\big(\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi)\big)\right),\\
v_t &= \frac{y_t^3}{6} + \frac{2k^2}{|\alpha|}\operatorname{cn}^2(\sigma\varphi) y_t - \frac{4k^2}{\sigma\alpha}\operatorname{cn}(\sigma\varphi)\big(\operatorname{sn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi)\big)\\
&\quad + \frac{2k^2}{\sigma\alpha}\left(\frac{2}{3}\operatorname{cn}(\sigma\varphi_t)\operatorname{dn}(\sigma\varphi_t)\operatorname{sn}(\sigma\varphi_t) - \frac{2}{3}\operatorname{cn}(\sigma\varphi)\operatorname{dn}(\sigma\varphi)\operatorname{sn}(\sigma\varphi) + \frac{1-k^2}{3k^2}\sigma t\right.\\
&\quad \left. + \frac{2k^2-1}{3k^2}(\operatorname{E}(\sigma\varphi_t) - \operatorname{E}(\sigma\varphi))\right).
\end{aligned}$$

If $\lambda \in C_2$, then

$$\begin{aligned}
x_t &= \frac{2\sigma \operatorname{sgn} c}{\alpha k}(\operatorname{dn}(\sigma\psi_t) - \operatorname{dn}(\sigma\psi)),\\
y_t &= \frac{k^2-2}{k^2}\operatorname{sgn}\alpha t + \frac{2\sigma}{\alpha k}(\operatorname{E}(\sigma\psi_t) - \operatorname{E}(\sigma\psi)),\\
z_t &= -\frac{x_t y_t}{2} - \frac{2\sigma \operatorname{sgn} c \operatorname{dn}(\sigma\psi)}{\alpha k} y_t + \frac{2\operatorname{sgn} c}{|\alpha|}\big(\operatorname{cn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \operatorname{cn}(\sigma\psi)\operatorname{sn}(\sigma\psi)\big),\\
v_t &= \frac{4}{\sigma\alpha k}\left(\frac{1}{3}\operatorname{cn}(\sigma\psi_t)\operatorname{dn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \frac{1}{3}\operatorname{cn}(\sigma\psi)\operatorname{dn}(\sigma\psi)\operatorname{sn}(\sigma\psi)\right.\\
&\quad \left. - \frac{1-k^2}{3k^3}\sigma t - \frac{k^2-2}{6k^2}(\operatorname{E}(\sigma\psi_t) - \operatorname{E}(\sigma\psi))\right) + \frac{y_t^3}{6} + \frac{2y_t}{|\alpha|k^2}\operatorname{dn}^2(\sigma\psi)\\
&\quad - \frac{4}{\sigma\alpha k}\operatorname{dn}(\sigma\psi)\big(\operatorname{cn}(\sigma\psi_t)\operatorname{sn}(\sigma\psi_t) - \operatorname{cn}(\sigma\psi)\operatorname{sn}(\sigma\psi)\big).
\end{aligned}$$

If $\lambda \in C_3$, then

$$x_t = \frac{2\sigma \operatorname{sgn} c}{\alpha}\left(\frac{1}{\cosh(\sigma\varphi_t)} - \frac{1}{\cosh(\sigma\varphi)}\right),$$
$$y_t = \frac{2\sigma}{\alpha}(\tanh(\sigma\varphi_t) - \tanh(\sigma\varphi)) - \operatorname{sgn}\alpha t,$$
$$z_t = -\frac{x_t y_t}{2} - \frac{2\sigma \operatorname{sgn} c}{\alpha\cosh(\sigma\varphi)} y_t + 2\frac{\operatorname{sgn} c}{|\alpha|}\left(\frac{\tanh(\sigma\varphi_t)}{\cosh(\sigma\varphi_t)} - \frac{\tanh(\sigma\varphi)}{\cosh(\sigma\varphi)}\right),$$
$$v_t = \frac{2}{3\sigma\alpha}\left(\tanh(\sigma\varphi_t) - \tanh(\sigma\varphi) + 2\frac{\tanh(\sigma\varphi_t)}{\cosh^2(\sigma\varphi_t)} - 2\frac{\tanh(\sigma\varphi)}{\cosh^2(\sigma\varphi)}\right) + \frac{y_t^3}{6}$$
$$+ \frac{2y_t}{|\alpha|\cosh^2\varphi} - \frac{4}{\sigma\alpha\cosh(\sigma\varphi)}\left(\frac{\tanh(\sigma\varphi_t)}{\cosh(\sigma\varphi_t)} - \frac{\tanh(\sigma\varphi)}{\cosh(\sigma\varphi)}\right).$$

5.6. Parametrization of extremal trajectories for C_4, C_5 and C_6. If $\lambda \in C_4$, then $\dot\theta = 0$, $\cos\theta = \operatorname{sgn}\alpha$ and the Hamiltonian system is easy to integrate:

$$x_t = 0, \qquad y_t = t \operatorname{sgn}\alpha, \qquad z_t = 0, \qquad v_t = \frac{t^3}{6}\operatorname{sgn}\alpha.$$

For $\lambda \in C_5$ we have $\cos\theta = -\operatorname{sgn}\alpha$, so that

$$x_t = 0, \qquad y_t = -t \operatorname{sgn}\alpha, \qquad z_t = 0, \qquad v_t = -\frac{t^3}{6}\operatorname{sgn}\alpha.$$

Let $\lambda \in C_6$; then $\alpha = 0$ and $c \neq 0$. Hence $\ddot\theta_t = 0$, so that $\theta_t = ct + \theta$, where $c = \text{const}$ and $\theta = \text{const}$. This gives us

$$x_t = \frac{\cos(ct+\theta) - \cos\theta}{c}, \qquad y_t = \frac{\sin(ct+\theta) - \sin\theta}{c}, \qquad z_t = \frac{ct - \sin(ct)}{2c^2},$$
$$v_t = -\frac{2c\cos\theta t - 4\sin(ct+\theta) + \sin(2ct+\theta)}{4c^3}.$$

For $\lambda \in C_7$ we have $\alpha = c = 0$ and $\theta_t \equiv \theta = \text{const}$, so that

$$x_t = -t\sin\theta, \qquad y_t = t\cos\theta, \qquad z_t = 0, \qquad v_t = \frac{t^3}{6}\cos\theta.$$

Note that normal extremal trajectories of the cases C_4 and C_5 coincide with the abnormal trajectories in (4.3), so the latter are not strictly abnormal.

If the pendulum (5.8) oscillates with subcritical energy $E < |\alpha|$ (the case C_1), then the elastics (x_t, y_t) have inflection points and are said to be *inflection* elastics (Figs. 5–7). If it rotates with supercritical energy $E > |\alpha|$ (the case C_2), then they have no inflection points and are called *noninflection elastics* (Fig. 8). Finally, if a pendulum moves with critical energy $E = |\alpha|$ (the case C_3), then the corresponding elastic is said to be *critical* (Fig. 9). For a pendulum rotating in weightlessness (the case C_6), the projections of extremal trajectories onto the (x, y)-planes are circles. For a pendulum at rest (the cases C_4, C_5 and C_7) these projections are straight lines.

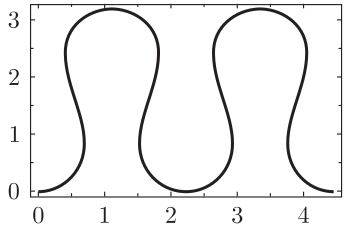

Figure 5. An inflection elastic.

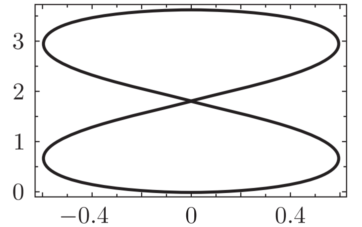

Figure 6. An inflection elastic.

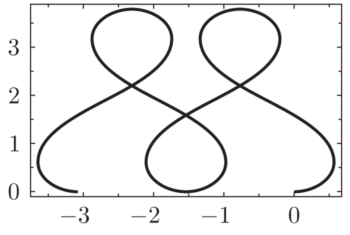

Figure 7. An inflection elastic.

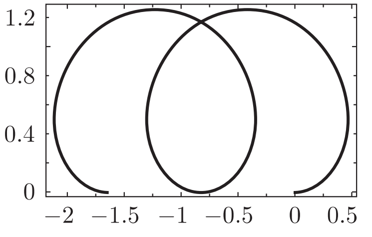

Figure 8. A noninflection elastic.

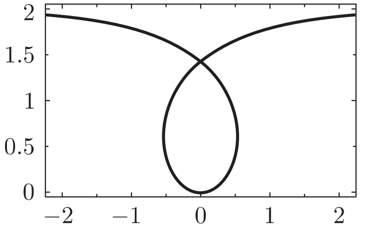

Figure 9. A critical elastic.

5.7. The exponential map. The family of all extremal trajectories is described by the exponential map

$$\begin{gathered}\operatorname{Exp}\colon C \times \mathbb{R}_+ \to M, \qquad \operatorname{Exp}(\lambda, t) = q_t,\\ \lambda = (\theta, c, \alpha) \in C, \qquad t \in \mathbb{R}_+, \qquad q_t \in M.\end{gathered}$$

It takes a pair (λ, t) consisting of the initial value of the vector of costate variables $\lambda \in C$ and time $t \in \mathbb{R}_+$ to the terminal point of the corresponding extremal trajectory q_t. In the previous subsections we obtained explicit expressions for the exponential map in terms of elementary and Jacobi functions.

Now we investigate discrete symmetries of the exponential map and on their basis find estimates for the cut time on extremal trajectories.

§ 6. Discrete symmetries of the exponential map

6.1. Reflections of the pendulum direction field. The subsystem (5.1)–(5.3) for the constate variables of the normal Hamiltonian system reduces to the pendulum equation (5.8). Obviously, the following reflections ε^i preserve the direction field of this system:

$$\varepsilon^1\colon (\theta, c, \alpha) \mapsto (\theta, -c, \alpha), \qquad \varepsilon^2\colon (\theta, c, \alpha) \mapsto (-\theta, c, \alpha),$$
$$\varepsilon^3\colon (\theta, c, \alpha) \mapsto (-\theta, -c, \alpha), \qquad \varepsilon^4\colon (\theta, c, \alpha) \mapsto (\theta + \pi, c, -\alpha),$$
$$\varepsilon^5\colon (\theta, c, \alpha) \mapsto (\theta + \pi, -c, -\alpha), \qquad \varepsilon^6\colon (\theta, c, \alpha) \mapsto (-\theta + \pi, c, -\alpha),$$
$$\varepsilon^7\colon (\theta, c, \alpha) \mapsto (-\theta + \pi, -c, -\alpha).$$

The reflection ε^1, ε^2, ε^5 and ε^6 reverse time on trajectories, while ε^3, ε^4 and ε^7 respect the direction of time. These reflections generate the symmetry group of a parallelepiped, $G = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6, \varepsilon^7\}$, with multiplication table shown in Table 1; we do not write out the entries below the main diagonal because G is an Abelian group.

Table 1. Multiplication in the group G

	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
ε^1	Id	ε^3	ε^2	ε^5	ε^4	ε^7	ε^6
ε^2		Id	ε^1	ε^6	ε^7	ε^4	ε^5
ε^3			Id	ε^7	ε^6	ε^5	ε^4
ε^4				Id	ε^1	ε^2	ε^3
ε^5					Id	ε^3	ε^2
ε^6						Id	ε^1
ε^7							Id

6.2. Reflections of trajectories of the pendulum equation. The action of the symmetries ε^i extends to the set of trajectories (with indicated direction of time) of the pendulum equation (5.8). Consider a smooth curve

$$\gamma = \{(\theta_s, c_s, \alpha) \mid s \in [0, t]\}$$

in the phase space of the pendulum, $S^1 \times \mathbb{R}^2$. We define the action of reflections on such curves (Fig. 10) as follows:

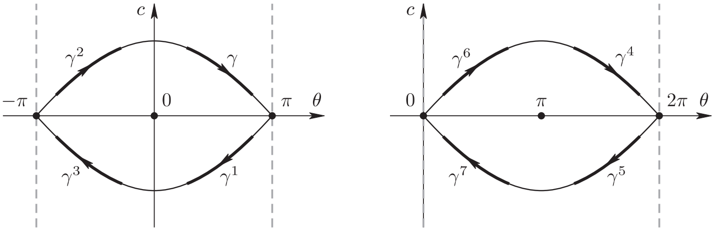

Figure 10. Reflections of trajectories of the pendulum.

$$
\begin{aligned}
&\varepsilon^1\colon \gamma \mapsto \gamma^1 = \{(\theta^1_s, c^1_s, \alpha^1) \mid s \in [0,t]\} = \{(\theta_{t-s}, -c_{t-s}, \alpha) \mid s \in [0,t]\},\\
&\varepsilon^2\colon \gamma \mapsto \gamma^2 = \{(\theta^2_s, c^2_s, \alpha^2) \mid s \in [0,t]\} = \{(-\theta_{t-s}, c_{t-s}, \alpha) \mid s \in [0,t]\},\\
&\varepsilon^3\colon \gamma \mapsto \gamma^3 = \{(\theta^3_s, c^3_s, \alpha^3) \mid s \in [0,t]\} = \{(-\theta_s, -c_s, \alpha) \mid s \in [0,t]\},\\
&\varepsilon^4\colon \gamma \mapsto \gamma^4 = \{(\theta^4_s, c^4_s, \alpha^4) \mid s \in [0,t]\} = \{(\theta_s + \pi, c_s, -\alpha) \mid s \in [0,t]\},\\
&\varepsilon^5\colon \gamma \mapsto \gamma^5 = \{(\theta^5_s, c^5_s, \alpha^5) \mid s \in [0,t]\} = \{(\theta_{t-s} + \pi, -c_{t-s}, -\alpha) \mid s \in [0,t]\},\\
&\varepsilon^6\colon \gamma \mapsto \gamma^6 = \{(\theta^6_s, c^6_s, \alpha^6) \mid s \in [0,t]\} = \{(-\theta_{t-s} + \pi, c_{t-s}, -\alpha) \mid s \in [0,t]\},\\
&\varepsilon^7\colon \gamma \mapsto \gamma^7 = \{(\theta^7_s, c^7_s, \alpha^7) \mid s \in [0,t]\} = \{(-\theta_s + \pi, -c_s, -\alpha) \mid s \in [0,t]\}.
\end{aligned}
$$

Lemma 1. *The reflections ε^i, $i = 1, \ldots, 7$, transform trajectories of the pendulum equation* (5.8) *into trajectories.*

Proof. This can be verified by direct differentiation. For example, for ε^4 and ε^7 we obtain

$$
\begin{aligned}
&\frac{d}{ds}\theta^4_s = \frac{d}{ds}(\theta_s + \pi) = \dot{\theta}_s = c^4_s,\\
&\frac{d}{ds}c^4_s = \frac{d}{ds}c_s = \dot{c}_s = -\alpha \sin\theta_s = \alpha^4 \sin(\theta^4_s + \pi) = -\alpha^4 \sin\theta^4_s,\\
&\frac{d}{ds}\theta^7_s = \frac{d}{ds}(-\theta_s + \pi) = -\dot{\theta}_s = -c_s = c^7_s,\\
&\frac{d}{ds}c^7_s = \frac{d}{ds}(-c_s) = -\dot{c}_s = \alpha \sin\theta_s = -\alpha^7 \sin(-\theta^7_s + \pi) = -\alpha^7 \sin\theta^7_s.
\end{aligned}
$$

Now from the 'vertical' subsystem (5.1)–(5.3) we can extend the action of the ε^i to solutions of the full Hamiltonian system of Pontryagin's maximum principle (5.1)–(5.7)

$$
\dot{\theta}_s = c_s, \qquad \dot{c}_s = -\alpha \sin\theta_s, \qquad \dot{\alpha} = 0, \qquad \dot{q}_s = -\sin\theta_s X_1(q_s) + \cos\theta_s X_2(q_s)
$$

in the following fashion:

$$
\varepsilon^i\colon \big\{(\theta_s, c_s, \alpha, q_s) \mid s \in [0,t]\big\} \mapsto \big\{(\theta^i_s, c^i_s, \alpha^i, q^i_s) \mid s \in [0,t]\big\}. \tag{6.1}
$$

The action of reflections on the curves (θ_s, c_s, α) has already been explicitly described in this subsection. Below we calculate the action of reflections on the elastics (x_s, y_s) and on the end-points of geodesics q_t.

6.3. Reflections of Euler elastics. Let $q_s = (x_s, y_s, z_s, v_s)$, $s \in [0,t]$, be a geodesic and let

$$
q^i_s = (x^i_s, y^i_s, z^i_s, v^i_s), \qquad s \in [0,t], \quad i = 1, \ldots, 7,
$$

be its ε^i-images. We shall describe the action of reflections on the elastics (x_s, y_s).

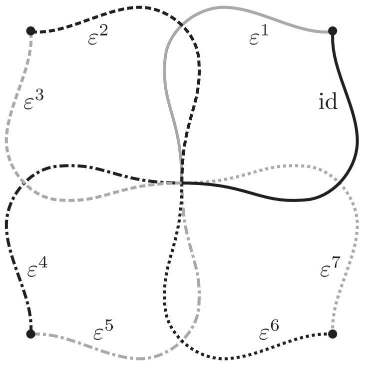

Figure 11. Reflections of Euler elastics.

Lemma 2. *The reflections* ε^i, $i = 1, \dots, 7$, *transform the elastics* (x_s, y_s) *as follows*:

$$
\begin{aligned}
x_s^1 &= x_t - x_{t-s}, & y_s^1 &= y_t - y_{t-s},\\
x_s^2 &= x_{t-s} - x_t, & y_s^2 &= y_t - y_{t-s},\\
x_s^3 &= -x_s, & y_s^3 &= y_s,\\
x_s^4 &= -x_s, & y_s^4 &= -y_s,\\
x_s^5 &= x_{t-s} - x_t, & y_s^5 &= y_{t-s} - y_t,\\
x_s^6 &= x_t - x_{t-s}, & y_s^6 &= y_{t-s} - y_t,\\
x_s^7 &= x_s, & y_s^7 &= -y_s.
\end{aligned}
$$

Proof. Using direct integration, for instance, for ε^6 we verify that

$$
\begin{aligned}
x_s^6 &= \int_0^s (-\sin(-\theta_{t-r} + \pi))\, dr = -\int_t^{t-s} (-\sin\theta_p)\, dp = x_t - x_{t-s},\\
y_s^6 &= \int_0^s \cos(-\theta_{t-r} + \pi)\, dr = \int_t^{t-s} \cos\theta_p\, dp = y_{t-s} - y_t.
\end{aligned}
$$

The action of the symmetries ε^i on elastics has the following geometric interpretations (Fig. 11):

ε^1 is the reflection of the elastic in its chord (the line segment connecting its end-points);

ε^3 is the reflection of the elastic in the y-axis;

ε^4 is the reflection of the elastic in the origin;

ε^7 is the reflection of the elastic in the x-axis;

other symmetries can be expressed as composites of the ones above, with the help of the multiplication table (see Table 1).

6.4. Reflections of the terminal points of geodesics.

Lemma 3. *The reflections* ε^i, $i = 1, \dots, 7$, *take the terminal points of the geodesics* $q_t = (x_t, y_t, z_t, v_t)$ *to the terminal points of the geodesics* $q_t^i = (x_t^i, y_t^i, z_t^i, v_t^i)$ *as follows*:

$$\begin{aligned}
&x_t^1 = x_t, && y_t^1 = y_t, && z_t^1 = -z_t, && v_t^1 = v_t - x_t z_t,\\
&x_t^2 = -x_t, && y_t^2 = y_t, && z_t^2 = z_t, && v_t^2 = v_t - x_t z_t,\\
&x_t^3 = -x_t, && y_t^3 = y_t, && z_t^3 = -z_t, && v_t^3 = v_t,\\
&x_t^4 = -x_t, && y_t^4 = -y_t, && z_t^4 = z_t, && v_t^4 = -v_t,\\
&x_t^5 = -x_t, && y_t^5 = -y_t, && z_t^5 = -z_t, && v_t^5 = -v_t + x_t z_t,\\
&x_t^6 = x_t, && y_t^6 = -y_t, && z_t^6 = z_t, && v_t^6 = -v_t + x_t z_t,\\
&x_t^7 = x_t, && y_t^7 = -y_t, && z_t^7 = -z_t, && v_t^7 = -v_t.
\end{aligned}$$

Proof. Lemma 2 gives us expressions for x_t^i and y_t^i. The expressions for the other variables are obtained by integration. For example, for ε^1 we have

$$\begin{aligned}
z_t^1 &= \frac{1}{2}\int_0^t (x_s^1 \dot y_s^1 + y_s^1 \dot x_s^1)\, ds = \frac{1}{2}\int_0^t \big((x_t - x_{t-s})\dot y_s^1 - (y_t - y_{t-s})\dot x_s^1\big)\, ds = -z_t,\\
v_t^1 &= \frac{1}{2}\int_0^t \dot y_s^1 \big((x_s^1)^2 + (y_s^1)^2\big)\, ds\\
&= \frac{y_t x_t^2}{2} + \frac{y_t^3}{2} - x_t \int_0^t x_s \dot y_s\, ds - y_t \int_0^t y_s \dot y_s\, ds + v_t\\
&= \frac{y_t x_t^3}{2} - \frac{x_t}{2}\int_0^t \big((x_s y_s)^{\cdot} + (x_s \dot y_s - y_s \dot x_s)\big)\, ds + v_t = v_t - x_t z_t.
\end{aligned}$$

For the other reflections ε^i the proof is similar.

6.5. Reflections as symmetries of the exponential map. We define the action of the reflections ε^i on the domain of the exponential map $C \times \mathbb{R}_+$ as the restriction of the action on the initial points of the pendulum trajectories defined in § 6.2:

$$\begin{gathered}
\varepsilon^i \colon C \times \mathbb{R}_+ \to C \times \mathbb{R}_+, \qquad \varepsilon^i(\theta, c, \alpha, t) = (\theta^i, c^i, \alpha^i, t),\\
(\theta^1, c^1, \alpha^1) = (\theta_t, -c_t, \alpha),\\
(\theta^2, c^2, \alpha^2) = (-\theta_t, c_t, \alpha),\\
(\theta^3, c^3, \alpha^3) = (-\theta, -c, \alpha),\\
(\theta^4, c^4, \alpha^4) = (\theta + \pi, c, -\alpha),\\
(\theta^5, c^5, \alpha^5) = (\theta_t + \pi, -c_t, -\alpha),\\
(\theta^6, c^6, \alpha^6) = (-\theta_t + \pi, c_t, -\alpha),\\
(\theta^7, c^7, \alpha^7) = (-\theta + \pi, -c, -\alpha).
\end{gathered}$$

The action of the reflections ε on the image M of the exponential map will be defined as the action on the terminal points of geodesics (see Lemma 3):

$$\varepsilon^i\colon M \to M, \qquad \varepsilon^i(q) = \varepsilon^i(x,y,z,v) = q^i = (x^i,y^i,z^i,v^i), \tag{6.2}$$
$$(x^1,y^1,z^1,v^1) = (x,y,-z,v-xz), \tag{6.3}$$
$$(x^2,y^2,z^2,v^2) = (-x,y,z,v-xz), \tag{6.4}$$
$$(x^3,y^3,z^3,v^3) = (-x,y,-z,v), \tag{6.5}$$
$$(x^4,y^4,z^4,v^4) = (-x,-y,z,-v), \tag{6.6}$$
$$(x^5,y^5,z^5,v^5) = (-x,-y,-z,-v+xz), \tag{6.7}$$
$$(x^6,y^6,z^6,v^6) = (x,-y,z,-v+xz), \tag{6.8}$$
$$(x^7,y^7,z^7,v^7) = (x,-y,-z,-v). \tag{6.9}$$

Since the actions of the ε^i on the domain $C \times \mathbb{R}_+$ and the image M of the exponential map are induced by the actions of the reflections (6.1) on trajectories of the Hamiltonian system, we have the following result.

Proposition 1. *For each $i = 1,\dots,7$ the reflection ε^i is a symmetry of the exponential map in the following sense*:

$$\varepsilon^i \circ \operatorname{Exp}(\theta,c,\alpha,t) = \operatorname{Exp} \circ\, \varepsilon^i(\theta,c,\alpha,t), \qquad (\theta,c,\alpha) \in C, \quad t \in \mathbb{R}_+.$$

§ 7. Maxwell points

A point q_t in a sub-Riemannian geodesic is called a *Maxwell point* if there exists another extremal trajectory $\tilde{q}_s \not\equiv q_s$ for which $\tilde{q}_t = q_t$, $t > 0$. It is known that a geodesic cannot be optimal past a Maxwell point (see [9]). In this section we calculate the Maxwell points corresponding to some reflections ε^i. On this basis we derive estimates for the cut time along extremal trajectories

$$t_{\mathrm{cut}}(\lambda) = \sup\bigl\{t > 0 \mid \operatorname{Exp}(\lambda,s) \text{ is optimal for } s \in [0,t]\bigr\}.$$

In the domain of Exp we introduce the Maxwell sets corresponding to the symmetries ε^i:

$$\begin{gathered} \mathrm{MAX}^i = \bigl\{(\lambda,t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \operatorname{Exp}(\lambda^i,t) = \operatorname{Exp}(\lambda,t)\bigr\}, \\ \lambda = (\theta,c,\alpha), \qquad \lambda^i = (\theta^i,c^i,\alpha^i) = \varepsilon^i(\lambda). \end{gathered} \tag{7.1}$$

7.1. Fixed point of symmetries in the image of the exponential map. By Proposition 1 the equality $\operatorname{Exp}(\lambda^i,t) = \operatorname{Exp}(\lambda,t)$ in the definition (7.1) of MAX^i is equivalent to $\varepsilon^i(q_t) = q_t$. The following description of fixed points of the reflections ε^i in the image of the exponential map immediately follows from the definition (6.2)–(6.9) of the action of reflections on M.

Lemma 4.

$$
\begin{array}{llcl}
1) & \varepsilon^1(q) = q & \iff & z = 0; \\
2) & \varepsilon^2(q) = q & \iff & x = 0; \\
3) & \varepsilon^3(q) = q & \iff & x^2 + z^2 = 0; \\
4) & \varepsilon^4(q) = q & \iff & x^2 + y^2 + v^2 = 0; \\
5) & \varepsilon^5(q) = q & \iff & x^2 + y^2 + z^2 + v^2 = 0; \\
6) & \varepsilon^6(q) = q & \iff & y^2 + (2v - xz)^2 = 0; \\
7) & \varepsilon^7(q) = q & \iff & y^2 + z^2 + v^2 = 0.
\end{array}
$$

The equalities $\varepsilon^i(q) = q$ describe $\mathbb{R}^4$ submanifolds of dimension from 3 through 0 of the state space. In this paper we content ourselves with investigating submanifolds of the maximum dimension 3 (corresponding to the Maxwell sets MAX^1 and MAX^2). We investigate lower-dimensional submanifolds elsewhere.

7.2. Fixed points of symmetries in the domain of the exponential map. In this subsection we calculate the solutions of the equations $\lambda^i = \lambda$ essential for the description of the Maxwell sets MAX^i (for $i = 1, 2$); see (7.1). Here and throughout, in $C_i \times \mathbb{R}_+$ we use the new variables defined as follows:

$$
(\lambda, t) \in (C_1 \cup C_3) \times \mathbb{R}_+ \implies \tau = \sigma \frac{\varphi + \varphi_t}{2}, \quad p = \frac{\sigma t}{2},
$$
$$
(\lambda, t) \in C_2 \times \mathbb{R}_+ \implies \tau = \sigma \frac{\varphi + \varphi_t}{2k}, \quad p = \frac{\sigma t}{2k}.
$$

Proposition 2. *For $(\lambda, t) \in C \times \mathbb{R}_+$ denote $\varepsilon^i(\lambda, t) = (\lambda^i, t)$. Then*

$$
1) \quad \lambda^1 = \lambda \iff \begin{cases} \operatorname{cn} \tau = 0 & \text{for } \lambda \in C_1, \\ \text{is impossible} & \text{for } \lambda \in C_2 \cup C_3 \cup C_6; \end{cases}
$$

$$
2) \quad \lambda^2 = \lambda \iff \begin{cases} \operatorname{sn} \tau = 0 & \text{for } \lambda \in C_1, \\ \operatorname{sn} \tau \operatorname{cn} \tau = 0 & \text{for } \lambda \in C_2, \\ \tau = 0 & \text{for } \lambda \in C_3, \\ 2\theta + ct = 2\pi n & \text{for } \lambda \in C_6. \end{cases}
$$

Proof. We only prove 1) since the proof of 2) is similar.

By the definition of the action of reflections on the domain of the exponential map (§ 6.5), $\lambda^1 = \lambda$ is tantamount to the equalities $\theta = \theta_t$ and $c = -c_t$.

If $\lambda \in C_2 \cup C_3 \cup C_6$, then c keeps sign along trajectories of the pendulum (5.8), so the equality $c = -c_t$ is impossible.

If $\lambda \in C_1$, then by the definition of the elliptic coordinates (§ 5.3)

$$
\begin{cases} \theta = \theta_t, \\ c = -c_t \end{cases} \iff \begin{cases} \operatorname{sn}(\sigma\varphi) = \operatorname{sn}(\sigma\varphi_t), \\ \operatorname{cn}(\sigma\varphi) = -\operatorname{cn}(\sigma\varphi_t) \end{cases} \iff \operatorname{cn} \tau = 0.
$$

In the last transition we have used addition formulae for Jacobi functions (see [24]).

7.3. A general description of the Maxwell sets corresponding to the reflections ε^1 and ε^2. The results of the previous two subsections can be summarized as follows.

Theorem 1. *Let* $(\lambda, t) \in \bigcup_{i=1}^3 (C_i \times \mathbb{R}_+)$ *and* $q_t = \operatorname{Exp}(\lambda, t)$. *Then*

$$1)\quad (\lambda, t) \in \mathrm{MAX}^1 \quad \Longleftrightarrow \quad \begin{cases} z_t = 0,\ \operatorname{cn}\tau \neq 0 & \text{for } \lambda \in C_1, \\ z_t = 0 & \text{for } \lambda \in C_2 \cup C_3; \end{cases}$$

$$2)\quad (\lambda, t) \in \mathrm{MAX}^2 \quad \Longleftrightarrow \quad \begin{cases} x_t = 0,\ \operatorname{sn}\tau \neq 0 & \text{for } \lambda \in C_1, \\ x_t = 0,\ \operatorname{sn}\tau \operatorname{cn}\tau \neq 0 & \text{for } \lambda \in C_2, \\ x_t = 0,\ \tau \neq 0 & \text{for } \lambda \in C_3. \end{cases}$$

7.4. A complete description of the Maxwell sets corresponding to the reflections ε^1 and ε^2. In this subsection we describe the roots on extremal trajectories of the equations $x_t = 0$ and $z_t = 0$. To do this we use the following expansions obtained on the basis of the expressions for extremal trajectories obtained previously (§§ 5.5, 5.6) and addition formulae for Jacobi functions.

If $\lambda \in C_1$, then

$$x_t = -\frac{4k\sigma}{\alpha} \frac{\operatorname{dn} p \operatorname{sn} p \operatorname{dn} \tau \operatorname{sn} \tau}{\Delta}, \qquad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau > 0, \tag{7.2}$$

$$z_t = \frac{4k}{|\alpha|} \frac{f_z(p) \operatorname{cn} \tau}{\Delta}, \qquad f_z(p) = \operatorname{dn} p \operatorname{sn} p + (p - 2\operatorname{E}(p)) \operatorname{cn} p. \tag{7.3}$$

If $\lambda \in C_2$, then

$$x_t = -\frac{4\sigma \operatorname{sgn} c}{\alpha k} \frac{\operatorname{cn} p \operatorname{sn} p \operatorname{cn} \tau \operatorname{sn} \tau}{\Delta}, \tag{7.4}$$

$$z_t = -\frac{4 \operatorname{sgn} c}{|\alpha| k^2} \frac{g_z(p) \operatorname{dn} \tau}{\Delta}, \tag{7.5}$$

$$g_z(p) = ((k^2 - 2)p + 2\operatorname{E}(p)) \operatorname{dn} p - k^2 \operatorname{sn} p \operatorname{sn} p. \tag{7.6}$$

If $\lambda \in C_3$, then

$$x_t = \frac{8\sigma \operatorname{sgn} c}{\alpha} \frac{\cosh p \cosh \tau}{\cosh(2p) + \cosh(2\tau)}, \tag{7.7}$$

$$z_t = \frac{8 \operatorname{sgn} c}{|\alpha|} \frac{\cosh \tau (p \cosh p - \operatorname{sh} p)}{\cosh(2p) + \cosh(2\tau)}. \tag{7.8}$$

If $\lambda \in C_6$, then

$$x_t = \frac{\cos(ct + \theta) - \cos\theta}{c}, \tag{7.9}$$

$$z_t = \frac{ct - \sin(ct)}{2c^2}. \tag{7.10}$$

The zeros of $f_z(p)$ and $g_z(p)$ are described in [9]. In Proposition 2.1 in [9] it is shown that all the positive zeros of $f_z(p,k)$ have the following form:

$$p_z^1 < p_z^2 < \dots < p_z^n < \cdots, \qquad p_z^n(k) \in (-K + 2Kn, K + 2Kn), \quad n \in \mathbb{N},$$

where $K(k)$ is the complete elliptic integral of the 1st kind. In Proposition 3.1 in [9] it was shown that $g_z(p)$ has no positive zeros. In view of this, from Theorem 1 and the expansions (7.2)–(7.10) we extract the following description of the Maxwell sets corresponding to the reflections ε^1 and ε^2 (we set $N_i = C_i \times \mathbb{R}_+$).

Theorem 2.

1) $\mathrm{MAX}^1 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = p_z^n(k),\ n \in \mathbb{N},\ \mathrm{cn}(\tau) \neq 0\}$;
2) $\mathrm{MAX}^1 \cap N_2 = \mathrm{MAX}^1 \cap N_3 = \mathrm{MAX}^1 \cap N_6 = \varnothing$;
3) $\mathrm{MAX}^2 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = 2Kn,\ n \in \mathbb{N},\ \mathrm{sn}(\tau) \neq 0\}$;
4) $\mathrm{MAX}^2 \cap N_2 = \{(\lambda, t) \in N_2 \mid p = Kn,\ n \in \mathbb{N},\ \mathrm{sn}(\tau)\,\mathrm{cn}(\tau) \neq 0\}$;
5) $\mathrm{MAX}^2 \cap N_3 = \varnothing$;
6) $\mathrm{MAX}^2 \cap N_6 = \{(\lambda, t) \in N_6 \mid tc = 2\pi n,\ \theta \neq \pi k,\ n, k \in \mathbb{Z}\}$.

7.5. Limit points of the Maxwell set. Let us introduce a subset of the closure of the Maxwell set:

$$\mathrm{CMAX} = \Big\{(\lambda, t) \in N \mid \exists \{\lambda_n, t_n\}, \{\lambda'_n, t_n\} \subset N :$$
$$\lambda_n \neq \lambda'_n,\ \mathrm{Exp}(\lambda_n, t_n) = \mathrm{Exp}(\lambda'_n, t_n),\ \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda'_n = \lambda\Big\}.$$

In Proposition 5.1 in [9] is was shown that if $(\lambda, t) \in \mathrm{CMAX}$, then the point

$$q_t = \mathrm{Exp}(\lambda, t)$$

is conjugate (to q_0) on the extremal trajectory $q_s = \mathrm{Exp}(\lambda, s)$ and the trajectory is not optimal for $s > t$.

Lemma 5.

1) *If* $(\lambda, t) \in N_1,\ p = p_z^n,\ n \in \mathbb{N},\ \mathrm{cn}\,\tau = 0,$ *then* $(\lambda, t) \in \mathrm{CMAX}$;
2) *if* $(\lambda, t) \in N_1,\ p = 2Kn,\ n \in \mathbb{N},\ \mathrm{sn}\,\tau = 0,$ *then* $(\lambda, t) \in \mathrm{CMAX}$;
3) *if* $(\lambda, t) \in N_2,\ p = Kn,\ n \in \mathbb{N},\ \mathrm{sn}\,\tau\,\mathrm{cn}\,\tau = 0,$ *then* $(\lambda, t) \in \mathrm{CMAX}$;
4) *if* $(\lambda, t) \in N_6,\ tc = 2\pi n,\ \theta = \pi k,\ n, k \in \mathbb{Z},$ *then* $(\lambda, t) \in \mathrm{CMAX}$.

Proof. We only prove 1) since the other proofs are similar. We fix $n \in \mathbb{N}$ and consider sequences of points $\{(\lambda_m^\pm, t_m) \mid m \in \mathbb{N}\}$ such that

$$p_m^\pm = p_z^n, \qquad \tau_m^\pm = \tau \pm \frac{1}{m}.$$

Then

$$\lambda_m^+ \neq \lambda_m^-, \qquad \lim_{m\to\infty} \lambda_m^+ = \lim_{m\to\infty} \lambda_m^- = \lambda.$$

Moreover,

$$(\lambda_m^-, t) = \varepsilon^1(\lambda_m^+, t), \qquad z_m^+(t) = z_m^-(t) = 0.$$

Hence it follows from Lemma 4 that

$$\operatorname{Exp}(\lambda_m^-, t) = \operatorname{Exp}(\lambda_m^+, t), \qquad m \in \mathbb{N},$$

so that $(\lambda, t) \in \mathrm{CMAX}$.

7.6. An estimate for the cut time. We introduce a function $t^1_{\mathrm{MAX}} \colon C \to (0, +\infty]$:

$$\lambda \in C_1 \quad \Longrightarrow \quad t^1_{\mathrm{MAX}} = \frac{\min(2p_z^1, 4K)}{\sigma},$$
$$\lambda \in C_2 \quad \Longrightarrow \quad t^1_{\mathrm{MAX}} = \frac{2Kk}{\sigma},$$
$$\lambda \in C_6 \quad \Longrightarrow \quad t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|},$$
$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \quad \Longrightarrow \quad t^1_{\mathrm{MAX}} = +\infty.$$

Theorem 2 and Lemma 5 yield the following global estimate for the cut time on extremal trajectories.

Theorem 3. *For each* $\lambda \in C$

$$t_{\mathrm{cut}}(\lambda) \leqslant t^1_{\mathrm{MAX}}(\lambda). \tag{7.11}$$

An analysis of the global structure of the exponential map shows that the estimate (7.11) is not sharp; however, with its help we can reduce the sub-Riemannian problem on the Engel group to systems of equations of the following form:

$$q_t(\lambda) = q_1, \qquad t \in (0, t^1_{\mathrm{MAX}}(\lambda)];$$

this is similar to the Euler elastic problem (see [25]). Thanks to this a software program calculating approximately the global solution to the sub-Riemannian problem on the Engel group can be designed, which can be used for a program of approximate solution of the control problem for general nonlinear systems with 2-dimensional control in a 4-dimensional space. These undertakings are the subject of a forthcoming paper.

§ 8. Conclusion

In this paper we have looked at the nilpotent sub-Riemannian problem on the Engel group. For this problem

- we have calculated the extremal trajectories;
- we have described the symmetries of the exponential map and the corresponding Maxwell points;
- we have obtained a global upper estimate for the cut time on extremal trajectories.

In the future we are intending to investigate the optimality of extremal trajectories and to design a program for calculating optimal trajectories in the sub-Riemannian problem on the Engel group; relying on the method of nonlinear approximation we shall also apply our results to the control problem for general non-linear systems with 2-dimensional control in a 4-dimensional space.

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