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CONTROLLABILITY OF HYPERSURFACE AND SOLVABLE INVARIANT SYSTEMS

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ABSTRACT. This paper deals with affine invariant control systems on Lie groups. Controllability conditions for hypersurface systems and for systems on solvable simply connected Lie groups are obtained. A lower bound of the number of controlled vector fields necessary to achieve controllability on simply connected Lie groups is given.

1. INTRODUCTION

In this paper we study controllability properties of invariant control systems on Lie groups. The basic controllability results for these systems were obtained by Jurdjevic and Sussmann [1]. Jurdjevic and Kupka [2] introduced a method of Lie saturation and applied it to obtain sufficient controllability conditions for invariant systems on simple and semisimple Lie groups. See also papers by Gauthier and Bornard [3], Silva Leite and Crouch [4], El Assoudi and Gauthier [5]. For recent results on controllability of invariant systems on Lie groups see, e.g., the papers by Bravo and Martin [6], Lovric [7], Ayala and Tirao [8], Enos [9], Sachkov [10], [11].

The structure of this paper is as follows.

In Sec. 2 the basic definitions are recalled, the problem is state, and the main results of the paper are presented.

In Sec. 3 the hypersurface invariant systems are studied. Invariant hypersurface systems were studied by Bravo and Martin; in [6] these authors give a criterion for controllability of a hypersurface invariant system under the additional condition that the Lie algebra generated by the controlled vector fields is ideal. This result is generalized and a controllability test for a general hypersurface invariant system is given. This test is applied then

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to obtain the necessary controllability conditions for invariant systems on simply connected Lie groups.

In Sec. 4 a controllability test for affine invariant systems on some subclass of simply connected solvable Lie groups is proved. Then a lower bound of the number of controlled vector fields is obtained which is necessary for achieving controllability on a simply connected Lie group.

Finally, in Sec. 5 two examples and some generalizations of the results presented in this paper are discussed.

2. Notation, definitions, and the main results

Suppose that G is a connected Lie group, L is its Lie algebra, and A, $B_1, \dots, B_m$ are right-invariant vector fields on G, i.e., $A, B_1, \dots, B_m \in L$. The *affine invariant system* on G is a subset of L of the form

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid \forall i \ u_i \in \mathbb{R} \right\}.$$

The *attainable set* $\mathbb{A}$ for the system Γ is a subsemigroup of G generated by the set $\left\{ \exp(tX) \mid X \in \Gamma,\ t \in \mathbb{R}_+ \right\}$. The system Γ is said to be *controllable* if $\mathbb{A} = G$.

The aim of this paper is to characterize the controllability of the system Γ in terms of the Lie group G, Lie algebra L, and vector fields A, $B_1, \dots, B_m$.

For any subset $l \subset L$, we denote by $\operatorname{Lie}(l)$ the Lie algebra generated by l.

We denote by L_0 the Lie algebra $\operatorname{Lie}(B_1, \dots, B_m)$, and by G_0 the corresponding connected subgroup of G.

For any $C \in L$ and $a \in G$ we denote by $C(a)$ the value of the right-invariant vector field C at the point a, and $l(a) = \left\{ X(a) \mid X \in l \right\}$ for any $l \subset L$.

The closure of the set P is denoted by $\overline{P}$. If M is a smooth manifold and $a \in M$, we denote by $\mathrm{T}_a M$ the tangent space of M at a.

We will consider derived series of the Lie algebra L:

$$L^{(1)} = \left[L, L\right], \quad L^{(2)} = \left[L^{(1)}, L^{(1)}\right], \quad \dots$$

The main results of the paper are as follows.

Theorem 1. *Suppose that L_0 has codimension one in L.*
(1) *If G_0 is closed in G, then Γ is controllable iff $A \notin L_0$ and $G/G_0 = S^1$.*
(2) *If G_0 is not closed in G, then Γ is controllable iff $A \notin L_0$.*

Theorem 1 generalizes the similar criterion obtained by Bravo and Martin under the additional assumption that L_0 is an ideal of L ([6], Proposition 3.3).

Theorem 2. *Let G be simply connected and solvable, and all operators $\operatorname{ad} x$, $x \in L$, have only real eigenvalues. Then Γ is controllable iff $L_0 = L$.*

Theorem 2 is a generalization of the similar result of Bravo and Martin for nilpotent systems ([6], Theorem 3.6).

3. Hypersurface systems

The affine system Γ is called a *hypersurface* if the Lie subalgebra L_0 has codimension one in L.

In this section we prove Theorem 1 which gives a controllability test for hypersurface invariant systems. This theorem is applied then to obtain the necessary controllability conditions for invariant systems on simply connected Lie groups (Corollaries 3.1–3.3).

3.1. Preliminary lemmas. Suppose that the Lie algebra L_0 has codimension one in L and the subgroup G_0 is closed in G. Then the right coset space G/G_0 is a smooth manifold, and the projection π is given by

$$\pi : G \to G/G_0, \qquad \pi : a \mapsto G_0 a.$$

For $a \in G$ we will consider the corresponding differential π_* and pull-back π^*:

$$\pi_*\big|_a : \mathrm{T}_a G \to \mathrm{T}_{\pi(a)} G/G_0, \qquad \pi^*\big|_a : \mathrm{T}^*_{\pi(a)} G/G_0 \to \mathrm{T}^*_a G.$$

The manifold G/G_0 is one-dimensional, and therefore it is either S^1 or $\mathbb{R}$. (To be more precise, G/G_0 is diffeomorphic to S^1 or $\mathbb{R}$, but this simplification is not essential.) We will denote by τ a differential form on G/G_0 coinciding with $d\varphi$ in the case $G/G_0 = S^1$ and with dx in the case $G/G_0 = \mathbb{R}$. Then $\pi^*\tau$ is a differential form on G which will be denoted by ω.

Lemma 3.1. *Let* $\dim L_0 = \dim L - 1$ *and* $\overline{G}_0 = G_0$. *Then* $\operatorname{Ker} \omega|_a = L_0(a)$ *for any* $a \in G$.

Proof. First of all, it is easy to see that

$$\forall a \in G, \qquad \operatorname{Ker} \pi_*|_a = \mathrm{T}_a(G_0 a) = L_0(a). \tag{1}$$

Then we consider the diagram

$$\begin{array}{ccc} \mathrm{T}_a G & \xrightarrow{\omega|_a} & \mathbb{R} \\ \big\downarrow \pi_*|_a & & \big\| \\ \mathrm{T}_{\pi(a)} G/G_0 & \xrightarrow{\tau|_{\pi(a)}} & \mathbb{R} \end{array}$$

This diagram is commutative, i.e.,

$$\omega|_a = \tau|_{\pi(a)} \circ \pi_*|_a. \tag{2}$$

But the space $\mathrm{T}_{\pi(a)} G/G_0$ is one-dimensional and that is why $\tau|_{\pi(a)}$ is an isomorphism. So, in view of (2), we have

$$\operatorname{Ker} \omega|_a = \operatorname{Ker} \pi_*|_a.$$

And then it follows from (1) that $\operatorname{Ker} \omega|_a = L_0(a)$. □

Lemma 3.2. *Suppose that* $\dim L_0 = \dim L - 1$, $\overline{G}_0 = G_0$, *and* $A \notin L_0$. *Then either* $\omega(X(a)) > 0$ *for all* $a \in G$, $X \in \Gamma$ *or* $\omega(X(a)) < 0$ *for all* $a \in G$, $X \in \Gamma$.

Proof. Consider the function $\Phi : G \times L \to \mathbb{R}$ defined as follows:

$$\forall a \in G, \ X \in L, \qquad \Phi(a, X) = \omega\big(X(a)\big).$$

Note that Φ is continuous on $G \times L$. To prove this lemma, we have to show that Φ has a constant sign on $G \times \Gamma$.

First we show that

$$\forall a \in G, \ B \in L_0, \qquad \Phi(a, B) = 0. \tag{3}$$

Take any $a \in G$ and $B \in L_0$. We have $B(a) \in L_0(a)$. By Lemma 3.1, $B(a) \in \operatorname{Ker} \omega|_a$, and relation (3) is proved.

Now we show that

$$\forall a \in G, \ X \in \Gamma, \qquad \Phi(a, X) \neq 0. \tag{4}$$

Let $a \in G$ and $X \in \Gamma$. Then $X = A + \sum_{i=1}^m u_i B_i$ for some real $u_1, \dots, u_m$, and therefore

$$\Phi(a, X) = \omega\left(A(a) + \sum_{i=1}^m u_i B_i(a)\right) = \omega\big(A(a)\big).$$

(We used here relation (3).) Next, $A \notin L_0$, and therefore $A(a) \notin L_0(a)$. Then, by Lemma 3.1, we have $\omega\big(A(a)\big) \neq 0$. Inequality (4) is proved.

Thus Φ is a continuous function which is nonvanishing on the arcwise connected set $G \times \Gamma$. That is why Φ has a constant sign on $G \times \Gamma$. □

Lemma 3.3. *Suppose that* $p \in \overline{\mathbb{A}}$. *Then*

(1) $G_0 p \subset \overline{\mathbb{A}}$,

(2) $\exp(tA)p \in \overline{\mathbb{A}}$ *for all* $t \geq 0$.

Proof. (1) easily follows from Lemma 6.4 [1], and (2) is obvious. □

Lemma 3.4. *Let* $\dim L_0 = \dim L - 1$, $\overline{G}_0 = G_0$, $G/G_0 = S^1$, *and* $A \notin L_0$. *Then* $\overline{\mathbb{A}} = G$.

Proof. According to Lemma 3.2, $\omega\big(A(a)\big)$ has the same sign for all $a \in G$. We will suppose that $\omega\big(A(a)\big) > 0$ for all $a \in G$. (In the case of the negative sign the proof is similar.)

We will show that $\pi(\mathbb{A}) = S^1$. Suppose that $\pi(\mathbb{A}) \neq S^1$.

(a) The attainable set $\mathbb{A}$ is arcwise connected (see [1], Lemma 4.4), and that is why $\pi(\mathbb{A})$ is arcwise connected too. Furthermore, the identity element $e \in G$ belongs to $\mathbb{A}$, and, hence $\pi(e) = 0 \in S^1$ belongs to $\pi(\mathbb{A})$. Thus $\pi(\mathbb{A})$ is a proper arcwise connected subset of the circle S^1 containing 0. That is why it is an interval of the form

$$\pi(\mathbb{A}) = |\alpha_{\min}; \alpha_{\max}| \tag{5}$$

for some numbers $\alpha_{\min} \leq 0 \leq \alpha_{\max}$, $\alpha_{\max} - \alpha_{\min} \leq 2\pi$, where $|$ stands for one of the brackets, [or].

(b) Choose any point $p \in \pi^{-1}(\alpha_{\max})$. Now we shall show that $p \in \overline{\mathbb{A}}$. It is easy to see that $\alpha_{\max} > 0$ and that there exists $\varepsilon > 0$ such that $(\alpha_{\max} - \varepsilon; \alpha_{\max}) \subset \pi(\mathbb{A})$. Take any $\alpha \in (\alpha_{\max} - \varepsilon; \alpha_{\max})$. We have $\alpha \in \pi(\mathbb{A})$, and therefore there exists a point $q \in \pi^{-1}(\alpha) \bigcap \mathbb{A}$. By Lemma 3.3, $G_0 q \subset \overline{\mathbb{A}}$. But $G_0 q = \pi^{-1}(\alpha)$, and so we have proved that

$$\exists \varepsilon > 0 \ \forall \alpha \in (\alpha_{\max} - \varepsilon; \alpha_{\max}), \qquad \pi^{-1}(\alpha) \subset \overline{\mathbb{A}}. \tag{6}$$

Let $V \subset G$ be any neighborhood of the point p. The projection π is an open map, and that is why $\pi(V) \subset S^1$ is a neighborhood of the point $\pi(p) = \alpha_{\max}$. Therefore there exists a point

$$\alpha \in (\alpha_{\max} - \varepsilon; \alpha_{\max}) \bigcap \pi(V).$$

On the one hand, $\alpha \in (\alpha_{\max} - \varepsilon; \alpha_{\max})$, and therefore (by (6)) we have

$$\pi^{-1}(\alpha) \subset \overline{\mathbb{A}}. \tag{7}$$

On the other hand, $\alpha \in \pi(V)$, and therefore

$$\pi^{-1}(\alpha) \bigcap V \neq \emptyset. \tag{8}$$

Then it follows from (7) and (8) that $V \bigcap \overline{\mathbb{A}} \neq \emptyset$. We have proved that any neighborhood of the point p intersects $\overline{\mathbb{A}}$. That is why $p \in \overline{\mathbb{A}}$.

(c) Now we come to a contradiction by showing that the set $\pi(\mathbb{A})$ contains numbers arbitrarily close to $\alpha_{\max}$ and greater than $\alpha_{\max}$.

Consider the curves $a(t) = \exp(At)p$ and $\alpha(t) = \pi\big(a(t)\big)$. We have $p \in \overline{\mathbb{A}}$, and therefore (by Lemma 3.3)

$$\{\, a(t) \mid t \geq 0 \,\} \subset \overline{\mathbb{A}}. \tag{9}$$

Furthermore, we have

$$\dot{\alpha}(0) = (d/dt)\big|_{t=0} \pi\big(\exp(At)p\big) = \pi_*\big(A(p)\big).$$

But $\omega\big(A(p)\big) > 0$, i.e., $d\varphi\big(\pi_*(A(p))\big) > 0$. That is why $\alpha(t)$ is increasing in the neighborhood of $t = 0$. Therefore there exist $\delta_0 > 0$ and $t_0 > 0$ such that $\alpha(t_0) = \alpha_{\max} + \delta_0$. In view of (9), we have $\{\, a(t) \mid 0 \le t \le t_0 \,\} \subset \overline{\mathbb{A}}$. But $\pi(\overline{\mathbb{A}}) \subset \overline{\pi(\mathbb{A})}$, and therefore

$$\pi\big(\{\, a(t) \mid 0 \le t \le t_0 \,\}\big) = \big[\, 0; \alpha_{\max} + \delta_0 \,\big] \subset \overline{\pi(\mathbb{A})}.$$

It contradicts (5). This contradiction shows that

$$\pi(\mathbb{A}) = S^1. \tag{10}$$

Now we shall show that $\overline{\mathbb{A}} = G$. We choose a point $x \in G$. In view of (10), there exists a point $y \in G_0 x \bigcap \mathbb{A}$. It follows from Lemma 3.3 that $G_0 y \subset \overline{\mathbb{A}}$. But $G_0 y = G_0 x$, and therefore $G_0 x \subset \overline{\mathbb{A}}$. Since $x \in G$ is arbitrary, we have $\overline{\mathbb{A}} = G$. □

Let us recall that the *rank controllability condition* (see [1]) for the invariant system Γ has the form

$$\mathrm{Lie}\,(\Gamma) = L.$$

For a hypersurface system this condition can easily be verified due to the following statement.

Lemma 3.5. *Let* $\dim L_0 = \dim L - 1$. *Then the rank controllability condition for* Γ *holds iff* $A \notin L_0$.

Proof. If $A \notin L_0$, then $L = L_0 \oplus \mathbb{R}A$ and $\mathrm{Lie}\,(\Gamma) = L$. And if $A \in L_0$, then $\mathrm{Lie}\,(\Gamma) = L_0 \neq L$. □

Lemma 3.6. *If the rank controllability condition for* Γ *is satisfied and the attainable set* $\mathbb{A}$ *is dense in* G*, then* Γ *is controllable.*

Proof. If the rank controllability condition for Γ holds, then the connected subgroup of G, corresponding to Lie algebra $\mathrm{Lie}\,(\Gamma)$, coincides with G, and then the statement follows immediately from Lemma 6.3 [1]. □

3.2. Proof of Theorem 1. Now we use the results of the previous subsection and prove Theorem 1.

Proof. The rank controllability condition is necessary for controllability ([1], Theorem 7.1), and therefore, by Lemma 3.5, the condition $A \notin L_0$ is necessary in both cases (1) and (2).

Case 1. Let $\overline{G}_0 = G_0$.

The necessity of $G/G_0 = S^1$. The closed subgroup G_0 has codimension one in G, and so dim $G/G_0 = 1$. That is why either $G/G_0 = \mathbb{R}$ or $G/G_0 = S^1$. If $G/G_0 = \mathbb{R}$, then Lemma 3.2 implies that the projection $\pi : G \to \mathbb{R}$ is a monotone function along all trajectories of the system Γ. It contradics the controllability of Γ, and therefore $G/G_0 = S^1$.

Sufficiency. Let $A \notin L_0$ and $G/G_0 = S^1$.

The rank controllability condition is satisfied by Lemma 3.5. By Lemma 3.4, the attainable set $\mathbb{A}$ is dense in G. And then it follows from Lemma 3.6 that Γ is controllable.

Case 2. Let $\overline{G}_0 \neq G_0$.

Sufficiency. We denote $\overline{G}_0$ by H. It is easy to see that H is an abstract subgroup of G. But H is closed in G, and therefore it is a Lie subgroup of G. Furthermore, G_0 is connected, and therefore H is connected. We denote the Lie algebra of H by $L(H)$. We have $G_0 \subset H \subset G$, and therefore $L_0 \subset L(H) \subset L$. But L_0 has codimension one in L, and therefore either $L(H) = L_0$ or $L(H) = L$. If $L(H) = L_0$, then $H = G_0$, which contradics the nonclosedness of G_0. Thus $L(H) = L$ and H is a connected subgroup of G with the Lie algebra L. That is why $H = G$, i.e., $\overline{G}_0 = G$. But $\overline{\mathbb{A}} \supset G_0$ (it is exactly the statement of [1], Lemma 6.4), and therefore we have $\overline{\mathbb{A}} = G$. The rank controllability condition is satisfied (see Lemma 3.5), and it follows from Lemma 3.6 that Γ is controllable. □

3.3. The necessary conditions for simply connected Lie groups. In this subsection we use Theorem 1 and obtain the necessary controllability conditions for affine invariant systems on simply connected Lie groups.

Corollary 3.1. *Suppose that G is simply connected and L_0 has codimension one in L. Then Γ is not controllable.*

Proof. If G is simply connected, then its codimension one subgroup G_0 is closed. Furthermore, G is simply connected, and that is why G/G_0 is simply connected too. Thus $G/G_0 = \mathbb{R}$, and it follows from Theorem 1 that Γ is not controllable. □

Corollary 3.2. *Let G be simply connected. Suppose that there exists a codimension one subalgebra l of L containing L_0. Then Γ is not controllable.*

Proof. The system Γ can be extended to an affine system of the form

$$\Gamma_1 = \left\{ A + \sum_{i=1}^{m} u_i B_i + \sum_{i=m+1}^{k} u_i B_i \mid \forall i \ u_i \in \mathbb{R} \right\},$$

where $B_{m+1}, \ldots, B_k$ complement $B_1, \ldots, B_m$ to a basis of the subalgebra l. By Corollary 3.1, the system Γ_1 is not controllable, and therefore Γ is not controllable too. □

Corollary 3.3. *Let G be simply connected and its Lie algebra L satisfy the following condition*:

$$\left.\begin{array}{l}\text{any subalgebra } l_1 \subset L,\ l_1 \neq L, \text{ is contained}\\ \text{in some codimension one subalgebra } l_2.\end{array}\right\} \qquad (*)$$

Then the affine system Γ on G is controllable iff $L_0 = L$.

Proof. It is well known (see [1], Corollary 7.4), that $L_0 = L$ is sufficient for the controllability of Γ.

If $L_0 \neq L$, then, by condition $(*)$, we can find a codimension one subalgebra $l \subset L$ such that $L_0 \subset l$. Then l satisfies the conditions of Corollary 3.2 and Γ is not controllable. □

4. Solvable systems

In this section we prove Theorem 2 and obtain a lower estimate of a number of controlled vector fields necessary for controllability on a simply connected Lie group (Theorem 3).

4.1. Codimension one subalgebras.

Lemma 4.1. *If L is solvable and all operators $\operatorname{ad} x$, $x \in L$, have only real eigenvalues, then for any subalgebra $l_1 \subset L$, $l_1 \neq L$, there exists a subalgebra $l_2 \subset L$ such that $l_1 \subset l_2$ and $\dim l_2 = \dim l_1 + 1$.*

Proof. Consider the representation

$$\rho : l_1 \to \operatorname{End}(L/l_1),$$

defined as follows:

$$\forall x \in l_1, v \in L, \quad \rho(x)(v + l_1) = [x, v] + l_1,$$

where ρ is the quotient representation of the adjoint representation

$$\operatorname{ad} : l_1 \to \operatorname{End}(L).$$

All operators $\rho(x)$, $x \in l_1$, have only real eigenvalues, l_1 being a solvable Lie algebra, and therefore, by Lie's theorem, the complexification of ρ has a common eigenvector. The corresponding eigenvalue is real, and therefore we can find a real common eigenvector $v + l_1 \in L/l_1$, $v \notin l_1$, for all operators $\rho(x)$, $x \in l_1$:

$$\forall x \in l_1 \quad \rho(x)(v + l_1) = [x, v] + l_1 = \lambda(x) v + l_1, \quad \lambda(x) \in \mathbb{R}. \qquad (11)$$

Consider the linear space

$$l_2 = l_1 + \operatorname{span}(v).$$

It follows from (11) that

$$[l_1, v] \subset l_2,$$

and therefore l_2 is a Lie algebra. Obviously, we have $l_1 \subset l_2$ and $\dim l_2 = \dim l_1 + 1$. □

Lemma 4.2. *If L is solvable and all operators* $\operatorname{ad} x$, $x \in L$, *have only real eigenvalues, then condition* $(*)$ *holds.*

Proof. We use Lemma 4.1 iteratively and find that any subalgebra $l \subset L$, $l \neq L$, is contained in the chain of subalgebras $l = l_0 \subset l_1 \subset \ldots \subset l_{r-1} \subset l_r = L$, such that

$$\dim l_{k+1} = \dim l_k + 1 \quad \forall k = 0, \ldots, r-1.$$

Then l_{r-1} is the codimension one subalgebra containing the Lie algebra l. □

We say that the Lie algebra L has a *flag of ideals* if there exist ideals I_p, $I_{p-1}, \ldots, I_1$, I_0 in L such that

$$L = I_p \supset I_{p-1} \supset \ldots \supset I_1 \supset I_0 = \{0\}$$

and

$$\dim I_k = k \quad \forall k = 0, 1, \ldots, p.$$

Lemma 4.3. *Let L have a flag of ideals. Then*
(1) *L is solvable*,
(2) *all operators* $\operatorname{ad} x$, $x \in L$, *have only real eigenvalues*,
(3) *any subalgebra $l \subset L$ has a flag of ideals.*

Proof. Let $L = I_p \supset I_{p-1} \supset \ldots \supset I_1 \supset I_0 = \{0\}$ be a flag of ideals.
(1) For the derived series we have

$$L^{(k)} \subset I_{p-k} \quad \forall k = 0, 1, \ldots, p.$$

That is why $L^{(p)} = \{0\}$, i.e., L is solvable.
(2) We choose a base $e_1, \ldots, e_p$ in L such that

$$I_k = \operatorname{span}(e_1, \ldots, e_k) \quad \forall k = 0, 1, \ldots, p.$$

I_k are ideals in L, and therefore

$$\operatorname{ad} x\,(I_k) \subset I_k \quad \forall x \in L.$$

All operators $\operatorname{ad} x$, $x \in L$, have triangular matrices in the above base. That is why all eigenvalues of $\operatorname{ad} x$, $x \in L$, are real.
(3) Consider the linear spaces J_k, $k = 0, 1, \ldots, p$, defined as

$$J_k = l \bigcap I_k.$$

We have

$$[l, J_k] = [l, l \bigcap I_k] \subset [l, l] \bigcap [l, I_k] \subset l \bigcap [L, I_k] \subset l \bigcap I_k = J_k,$$

where J_k are ideals in l, and $l = J_p \supset J_{p-1} \supset \ldots \supset J_1 \supset J_0 = \{0\}$. We have

$$\dim J_{k+1} - \dim J_k = 1 \text{ or } 0 \qquad \forall k = 2, \ldots, p-1.$$

Thus, to obtain a flag of ideals in l, it is sufficient to exclude from the sequence $\{J_k\}$ all ideals J_k for which the above difference is equal to 0. □

4.2. Proof of Theorem 2. Now the proof of Theorem 2 follows immediately from Corollary 3.3 and Lemma 4.2.

Theorem 2 and Lemma 4.3 imply the following

Corollary 4.1. *Let G be a simply connected solvable Lie group and its Lie algebra L have a flag of ideals. Then Γ is controllable on G iff $L_0 = L$.*

Remark. Any nilpotent Lie algebra has a flag of ideals, and therefore the above controllability criterion applies to systems on simply connected nilpotent Lie groups.

4.3. Quotient systems. Let h be an ideal of L and H be the corresponding connected subgroup of G. Suppose that H is closed, and so G/H is a Lie group. We denote the projection from G onto G/H by π and its differential by π_*. We can correctly define the projection of the system Γ onto G/H:

$$\pi_*(\Gamma) = \{ \pi_* v \mid v \in \Gamma \} \subset L/h.$$

Note that the controllability of the system Γ on G implies the controllability of the system $\pi_*(\Gamma)$ on G/H.

The derived subalgebra $L^{(1)}$ is an ideal of L and for simply connected G its derived subgroup $G^{(1)}$ is closed. The above construction and Theorem 2 allow us to give the following necessary controllability condition.

Theorem 3. *Let G be simply connected. If the system Γ is controllable, then*

(1) $\pi_*(L_0) = L/L^{(1)}$,

(2) $m \geq \dim L - \dim L^{(1)}$.

Proof. (1) The Lie algebra $L/L^{(1)}$ is commutative, and therefore it obviously satisfies condition $(*)$. If Γ is controllable on G, then $\pi_*(\Gamma)$ is controllable on $G/G^{(1)}$. Then it follows from Theorem 2 that $\pi_*(L_0) = L/L^{(1)}$.

(2) The subalgebra $\pi_*(L_0)$ is commutative and is spanned by vectors $\pi_* B_1, \ldots, \pi_* B_m$, and therefore

$$m \geq \dim(\pi_*(L_0)) = \dim(L/L^{(1)}) = \dim L - \dim L^{(1)}. \qquad \square$$

Remark. This theorem implies that invariant systems on a simply connected Lie group G with nontrivial $G/G^{(1)}$ essentially differ from general nonlinear systems. It is well known that generically $m = 2$ is sufficient for global controllability. But Proposition 2 of Theorem 3 gives a lower bound ($\dim G/G^{(1)}$) for the number of controlled vector fields $B_1, \dots, B_m$ necessary to achieve controllability on simply connected G.

5. Examples

Example 1. Let $G = \mathrm{T}(n)$ be the Lie group of all $n \times n$ upper triangular matrices with positive diagonal entries. $\mathrm{T}(n)$ is connected, simply connected, and solvable. Its Lie algebra $L = \mathrm{t}(n)$ consists of all $n \times n$ upper triangular matrices. The derived subalgebra $L^{(1)}$ consists of all strictly upper triangular matrices, and $L/L^{(1)}$ is the Lie algebra of all diagonal $n \times n$ matrices.

By Theorem 2, the affine system Γ is controllable on $\mathrm{T}(n)$ if and only if $L_0 = L$, and, by Theorem 3, the controllability of Γ on $\mathrm{T}(n)$ can be achieved with no less than n controlled vector fields. This lower estimate is exact. For example, the system $\Gamma = \left\{ A + \sum_{i=1}^{n} u_i B_i \mid \forall i \ u_i \in \mathbb{R} \right\}$ with $B_i = E_{ii} + E_{i,i+1}$ for $i = 1, \dots, n-1$ and $B_n = E_{nn}$ is controllable on $\mathrm{T}(n)$. Really, it is easy to see that $\mathrm{Lie}\,(B_1, \dots, B_n) = \mathrm{t}(n)$.

(E_{ij} denotes the $n \times n$ matrix with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.)

Example 2. Let $G = \mathrm{E}(2)$ be the Euclidean group of motions of $\mathbb{R}^2$. $\mathrm{E}(2)$ is connected but not simply connected. It can be represented by 3×3 matrices of the form

$$\begin{pmatrix} c_{11} & c_{12} & b_1 \\ c_{21} & c_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = (c_{ij}) \in \mathrm{SO}(2), \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2,$$

where C is a rotation matrix and b is a translation vector. The corresponding matrix Lie algebra L is spanned by the matrices $A_1 = E_{13}$, $A_2 = E_{23}$, and $A_3 = E_{21} - E_{12}$. We have $L^{(1)} = \mathrm{span}(A_1, A_2)$ and $L^{(2)} = \{0\}$; therefore, L is solvable.

Consider the invariant system $\Gamma = \left\{ A_1 + uA_3 \mid u \in \mathbb{R} \right\}$.

Now we shall use the Lie saturation technique introduced by Jurdjevic and Kupka and show that the system Γ is controllable on $\mathrm{E}(2)$. (See [2] for definitions and details.) The Lie saturate of Γ will be denoted by $\mathrm{LS}(\Gamma)$.

We have $A_1, \pm A_3 \in \mathrm{LS}(\Gamma)$; that is why $\exp(s \,\mathrm{ad}\, A_3)A_1 \in \mathrm{LS}(\Gamma)$ for any $s \in \mathbb{R}$. But $\exp(s \,\mathrm{ad}\, A_3)A_1 = (\cos s)A_1 + (\sin s)A_2$. Consequently, $\mathrm{span}(A_1, A_2) \subset \mathrm{LS}(\Gamma)$; therefore, $\mathrm{LS}(\Gamma) = L$. Thus Γ is controllable on $\mathrm{E}(2)$.

Obviously, Γ can also be considered as an invariant system on the simply connected covering group of $\mathrm{E}(2)$. The above proof of controllability of Γ on

E(2) is purely algebraic; i.e., it does not use any global geometric properties of E(2). That is why Γ is controllable on the simply connected covering group of E(2) as well.

The spectrum of the operator $\operatorname{ad} A_3$ consists of $\pm i$ and 0. Therefore this example shows that the assumption of reality of the spectrum in Theorem 2 is essential. Detailed controllability conditions for invariant systems on solvable Lie groups without this assumption can be obtained with the use of the necessary conditions given in 3.3 and the Lie saturation technique. These results will be published in our forthcoming paper.

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