On invariant orthants of bilinear systems

Yu. L. Sachkov *
Program Systems Institute
Russian Academy of Sciences
152140 Pereslavl-Zalessky
Russia
email: sachkov@sys.botik.ru

September 8, 1997

Abstract. A characterization of $n \times n$ matrices A such that the corresponding linear vector field Ax has invariant orthants in $\mathbb{R}^n$ is obtained. This result is then applied to give necessary global controllability conditions for bilinear systems.

Key words and phrases. Controllability, bilinear systems, invariant domains, orthant, sign-symmetric matrices.

1991 *Mathematics Subject Classification.* 93B05.

1 Introduction

Bilinear systems form the simplest class of nonlinear control systems. They arise naturally as models in solving various applied problems (see, e.g., [2, 9]). There is a variety of results on global controllability of bilinear systems: [1, 3, 5, 6, 7, 8, 10, 11, 12, 13]; but this problem is far from complete solution.

In this paper we study conditions of existence of invariant orthants for bilinear systems

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u_i \in \mathbb{R}, \tag{1}$$

i.e., sufficient conditions for global noncontrollability of such systems. All invariant orthants are described, a constructive method of their enumeration is given, and their number is evaluated.

*This work was partially supported by the Russian Foundation for Fundamental Research, Projects 96-01-00805 and 97-1-1a/22. The author is a recipient of the 1997 Russian State Scientific Stipend for young researchers.

It turns out that system (1) has invariant orthants iff the matrices B_i, $i = 1, \dots, m$, are diagonal and the field Ax has invariant orthants. The search of these orthants is based upon two ideas. First, it is common knowledge that the positive orthant $\mathbb{R}^n_+$ is positive invariant for the field Ax iff all off-diagonal entries of the matrix A are nonnegative. Second, if the field Ax has an invariant orthant, then successive changes of coordinates $(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, -x_i, \dots, x_n)$ in $\mathbb{R}^n$ map this orthant onto $\mathbb{R}^n_+$. During this process we can keep track of signs of entries a_{ij} of the matrix A and obtain conditions of existence of invariant orthants in terms of sign combinations of a_{ij}. These conditions can conveniently be expressed in terms of some graph $\Gamma(A)$ that corresponds to the matrix A. An analogous method was applied by M. Hirsch [4] for studying limit properties of trajectories of dynamical systems.

This work has the following structure. In Sec. 2 we describe construction of the graph $\Gamma(A)$ for a sign-symmetric matrix A and the theorem of M. Hirsch that yields some combinatorial properties of such graphs. Then we present and proof a constructive version of this result — Theorem 2. In Sec. 3 we obtain conditions of existence of positive (negative) invariant orthants for the field Ax (Theorems 3, 4) in terms of the graph $\Gamma(A)$. In the generic case when all off-diagonal entries of the matrix A are nonzero, there are more simple tests (Theorem 5, this is a correction of the earlier author's results [13]). In Sec. 4 these results are applied to obtain the main propositions of the work — tests of existence of invariant orthants for system (1). In Sec. 5 we discuss the relation of the results obtained with the conjecture of V. Jurdjevic and I. Kupka on noncontrollability of the single-input system $\dot{x} = Ax + uBx$ in the case of symmetric matrices A and B.

Now we give some of the notation and definitions used in the sequel.

The set of indices $\Sigma_n = \{\, \sigma = (\sigma_1, \dots, \sigma_n) \mid \sigma_i = \pm 1 \ \forall i = 1, \dots, n \,\}$ will be used for parametrization of orthants, i.e., sets of the form $\mathbb{R}^n_\sigma = \{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \sigma_i \geq 0 \ \forall i = 1, \dots, n \,\}$.

A subset of the state space is called *positive* (*negative*) *invariant* for a vector field or a control system if all trajectories of the field or the system starting in this set (resp., its complement) do not leave it (resp., its complement) for all positive moments of time. A control system is called *globally controllable* if any points of its state space can be connected by a trajectory of the system (in the positive direction).

Remark. A system is globally controllable iff it has neither positive nor negative invariant sets (except the trivial ones — the whole state space and the empty set). Thus conditions of existence of nontrivial invariant sets are sufficient conditions of global noncontrollability.

2 Sign-symmetric matrices and their graphs

Definition. An $n \times n$ matrix $A = (a_{ij})$ is called *sign-symmetric* if $a_{ij}a_{ji} \geq 0$ for all $i, j = 1, \ldots, n$.

The role of this definition for this work is explained by Corollary 1 in the next section.

Construction. (M. Hirsch [4]) For any sign-symmetric $n \times n$ matrix A we construct the *graph* $\Gamma(A)$ by the following rule. The graph $\Gamma(A)$ has n vertices $1, 2, \ldots, n$. Its vertices i, j, $i \neq j$, are connected by the edge (i, j) iff at least one of the numbers a_{ij}, a_{ji} is nonzero. Throughout the paper we take into account only edges that connect *distinct* vertices of the graph $\Gamma(A)$; self loops are thus explicitly excluded from consideration. Every edge (i, j) is marked by the sign $+$ or $-$: if $a_{ij} \geq 0$ and $a_{ji} \geq 0$, then the sign $+$ is applied, and if $a_{ij} \leq 0$ and $a_{ji} \leq 0$, then we apply $-$ (there can be no other combinations of signs by virtue of sign-symmetry of A). The marked edges are called *positive* or *negative* depending on the sign $+$ or $-$. For the graph $\Gamma(A)$ we define the following function $s(i, j)$, $i, j = 1, \ldots, n$, $i \neq j$: $s(i, j) = 0$ if the vertices i, j are not connected by an edge in $\Gamma(A)$, $s(i, j) = 1$ for the positive, and $s(i, j) = -1$ for the negative edge (i, j) in the graph $\Gamma(A)$.

Definition. A loop (i.e., a closed path composed of edges) of a graph is called *even* (*odd*) if it contains an even (resp. odd) number of negative edges. A graph is said to *satisfy the even-loop property* if all its loops are even.

Remark. A loop $(i_1, i_2, \ldots, i_k, i_{k+1} = i_1)$ in the graph $\Gamma(A)$ is even if and only if $s(i_1, i_2)s(i_2, i_3) \ldots s(i_k, i_1) = 1$.

Theorem 1 (M. Hirsch [4]) *If a graph Γ satisfies the even-loop property, then there is a subset V of the set of its vertices such that:*

a) any negative edge of the graph Γ has exactly one vertex in V;

b) any positive edge of the graph Γ has either 0 or 2 vertices in V.

In [4] an algorithm of construction of the set V is presented as well.

Below in Theorem 2 we obtain a version of Theorem 1 that gives an explicit method of construction of the set V and evaluates the number of such sets for a fixed graph Γ. Our argument does not formally use Theorem 1 and may be considered as a new proof of this theorem.

Definition. Let a graph Γ satisfy the even-loop property and be connected (i.e., any two vertices of Γ can be connected by a path of its edges), and let v be any vertex of Γ. A vertex of the graph Γ is called *even* (*odd*) *with respect to* v if it can be connected with v by a path containing even (resp. odd) number of negative edges. The set of even (odd) with respect to v vertices will be denoted by V_v^+ (resp. V_v^-).

Remarks. 1) Parity of a vertex p with respect to a vertex v does not depend on a path connecting p and v: the number of negative edges in different paths connecting p and v have the same parity since Γ satisfies the even-loop property.

2) By connectedness of the graph Γ, any its vertex is either even or odd with respect to v, i.e., the set of all vertices of Γ can be represented in the form $V_v^+ \cup V_v^-$, and in view of the previous remark, $V_v^+ \cap V_v^- = \emptyset$.

3) For any vertex p of the graph Γ

$$
\begin{aligned}
p \in V_v^+ \quad &\Rightarrow \quad V_p^+ = V_v^+, \ V_p^- = V_v^-, \\
p \in V_v^- \quad &\Rightarrow \quad V_p^+ = V_v^-, \ V_p^- = V_v^+,
\end{aligned}
$$

since for $p \in V_v^+$ parities with respect to p and v coincide one with another, and for $p \in V_v^-$ a point even with respect to p is odd with respect to v, and vice versa.

Theorem 2 *Let a graph Γ satisfy the even-loop property.*

1. *If Γ is connected, then there are exactly two distinct subsets V of the set of all its vertices that satisfy conditions a), b) of Theorem 1. They have the form V_v^+ and V_v^-, where v is any vertex of Γ.*

2. *If Γ is not connected and has c connected components, then there are 2^c ways of choice of the set V by independent construction in every connected component as in item 1.*

Proof. Let the graph Γ be connected and v any its vertex.

We show that the both sets V_v^+, V_v^- satisfy conditions a), b) of Theorem 1. Indeed, vertices of every negative edge have distinct parities with respect to v, hence one of them is contained in V_v^+, and another in V_v^-. And vertices of every positive edge have the same parity with respect to v and consequently both belong either to V_v^+ or to V_v^-. Thus both sets V_v^+, V_v^- satisfy conditions of Theorem 1.

Now we show that there are no other such sets. Let V be any such set. If $V = \emptyset$, then all edges of Γ are positive, hence $V_v^- = \emptyset = V$. That is why suppose that $V \neq \emptyset$ and choose any vertex $p \in V$. By properties a), b) of Theorem 1, all vertices of Γ even with respect to p are contained in V, and all vertices odd with respect to p do not belong to V, i.e., $V = V_p^+$. By remark 3) before this theorem, $V_p^+ = V_v^+$ or V_v^-. Thus any set V that satisfies conditions a), b) of Theorem 1 is equal to V_v^+ or V_v^-.

The statement of the theorem for a connected graph Γ is proved, and for a disconnected one it is obvious. $\square$

3 Invariant orthants of linear vector fields

The following test of invariance of an orthant was proved in [13].

Lemma 1 *Let $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_n$, and let A be an $n \times n$ matrix. The orthant $\mathbb{R}^n_\sigma$ is positive (negative) invariant for the vector field Ax if and only if*

$$a_{ij}\sigma_i\sigma_j \geq 0 \ (\textit{resp.} \leq 0) \quad \forall i \neq j. \tag{2}$$

Corollary 1 *If the field Ax has a positive or negative invariant orthant, then the matrix A is sign-symmetric.*

Proof. By virtue of (2), $a_{ij}\sigma_i\sigma_j a_{ji}\sigma_j\sigma_i = a_{ij}a_{ji}\sigma_i^2\sigma_j^2 = a_{ij}a_{ji} \geq 0$. □

Corollary 2 *If there is an orthant positive and negative invariant for a field Bx, then the matrix B is diagonal.*

Proof. Let $B = (b_{ij})$; choose any $i \neq j$, $i, j = 1, \dots, n$, and show that $b_{ij} = 0$. By virtue of Lemma 1, we have $b_{ij}\sigma_i\sigma_j \geq 0$ and $b_{ij}\sigma_i\sigma_j \leq 0$, i.e., $b_{ij}\sigma_i\sigma_j b_{ij}\sigma_i\sigma_j = b_{ij}^2\sigma_i^2\sigma_j^2 = b_{ij}^2 \leq 0$, whence $b_{ij} = 0$. □

Construction. Assume that a graph Γ satisfies the even-loop property and V is any subset of the set of its vertices that satisfies conditions a), b) of Theorem 1. Then the *index of the graph* Γ *corresponding to the set* V is the set $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_n$ defined as follows: $\sigma_i = +1$ if $i \notin V$ and $\sigma_i = -1$ if $i \in V$.

Remark. The index is determined both by the graph Γ and by the set V. Theorem 2 shows that for a connected graph there are exactly two indices (distinguished by the common multiplier -1). For a disconnected graph with c connected components there are 2^c distinct indices, each being chosen independently for every connected component of Γ.

Theorem 3 *Let A be a sign-symmetric $n \times n$ matrix. The field Ax has positive invariant orthants iff its graph $\Gamma(A)$ satisfies the even-loop property. Then positive invariant orthants are $\mathbb{R}^n_\sigma$, where σ is any index of the graph $\Gamma(A)$; their number is equal to 2^c, where c is the number of connected components of the graph $\Gamma(A)$.*

Proof. Necessity. Suppose that the field Ax has a positive invariant orthant $\mathbb{R}^n_\sigma$, $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_n$. By Lemma 1, then inequalities (2) hold, that is why

$$\operatorname{sgn} a_{ij} = \sigma_i\sigma_j \text{ or } 0. \tag{3}$$

Choose any loop $(i_1, i_2, \dots, i_k, i_{k+1} = i_1)$ in the graph $\Gamma(A)$. Any pair of vertices (i_l, i_{l+1}), $l = 1, \dots, k$, is connected by an edge in $\Gamma(A)$, that is why $s(i_l, i_{l+1}) \neq 0$

(the function $s(i,j)$ that determines sign of the edge (i,j) was defined at the beginning of Sec. 2). Taking into account (3), we obtain $s(i_l, i_{l+1}) = \sigma_{i_l}\sigma_{i_{l+1}}$, $l = 1,\dots,k$. Consequently,

$$s(i_1,i_2)s(i_2,i_3)\dots s(i_k,i_1) = \sigma_{i_1}\sigma_{i_2}\sigma_{i_2}\sigma_{i_3}\dots\sigma_{i_k}\sigma_{i_1} = \sigma_{i_1}^2\sigma_{i_2}^2\dots\sigma_{i_k}^2 = 1,$$

i.e., the loop $(i_1, i_2, \dots, i_k, i_1)$ is even.

Sufficiency. Suppose that the graph $\Gamma(A)$ satisfies the even-loop property, V is the subset of the set of its vertices defined in Theorem 1, and σ is the corresponding index. We show that the orthant $\mathbb{R}^n_\sigma$ is positive invariant for the field Ax. By virtue of Lemma 1, it is sufficient to prove inequalities (2).

Choose any pair (i,j), $i \neq j$, $i,j = 1,\dots,n$.

If the vertices i,j are not connected by an edge in the graph $\Gamma(A)$, then $a_{ij} = 0$ and condition (2) holds.

Let the vertices i,j be connected by a positive edge. By the construction of $\Gamma(A)$, this means that $a_{ij} \geq 0$. Further, the positive edge (i,j) has either zero or two vertices in the set V. By definition of the index σ, if $i \notin V$, $j \notin V$, then $\sigma_i = \sigma_j = 1$; and if $i \in V$, $j \in V$, then $\sigma_i = \sigma_j = -1$. In any case $\sigma_i\sigma_j = 1$, and condition (2) holds.

Finally, suppose that the vertices i,j are connected by a negative edge. First, $a_{ij} \leq 0$. And second, the negative edge (i,j) has exactly one vertex in the set V. If $i \in V$, $j \notin V$, then $\sigma_i = -1$, $\sigma_j = 1$; and if $i \notin V$, $j \in V$, then $\sigma_i = 1$, $\sigma_j = -1$. That is why always $\sigma_i\sigma_j = -1$, and inequality (2) holds in this case as well.

Consequently, the orthant $\mathbb{R}^n_\sigma$ is positive invariant for the field Ax. The sufficiency is proved. The formula 2^c for the number of invariant orthants follows from Theorem 2. □

Remark. An orthant $\mathbb{R}^n_\sigma$ is negative invariant for the field Ax iff it is positive invariant for the field $-Ax$. Thus we obtain

Theorem 4 *Let A be a sign-symmetric $n \times n$ matrix. The field Ax has negative invariant orthants iff the graph $\Gamma(-A)$ of the opposite matrix $-A$ satisfies the even-loop property. Then negative invariant orthants are $\mathbb{R}^n_\sigma$, where σ is any index of the graph $\Gamma(-A)$; their number is equal to 2^c, where c is the number of connected components of the graph $\Gamma(-A)$ (or, which is the same, of the graph $\Gamma(A)$).*

Example 1. Let $A = (a_{ij})$ be any 4×4 matrix of the form

$$\begin{pmatrix} * & + & 0 & + \\ + & * & + & 0 \\ 0 & + & * & - \\ + & 0 & - & * \end{pmatrix},$$

i.e., $a_{12}, a_{21}, a_{14}, a_{41}, a_{23}, a_{32} > 0$, $a_{34}, a_{43} < 0$, $a_{13} = a_{31} = a_{24} = a_{42} = 0$, and diagonal entries are arbitrary. The corresponding graph $\Gamma(A)$ is given at Fig. 1. The only loop $(1, 2, 3, 4)$ is negative in both graphs $\Gamma(A)$ and $\Gamma(-A)$. That is why the field Ax has no invariant orthants (this may also be verified by direct inspection of 16 orthants in $\mathbb{R}^4$).

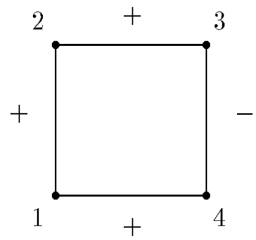

Fig. 1

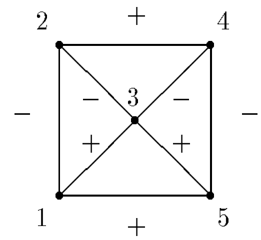

Fig. 2

Example 2. Let $A = (a_{ij})$ be any 5×5 matrix of the form

$$\begin{pmatrix} * & - & + & 0 & + \\ - & * & - & + & 0 \\ + & - & * & - & + \\ 0 & + & - & * & - \\ + & 0 & + & - & * \end{pmatrix},$$

i.e., $a_{13}, a_{31}, a_{15}, a_{51}, a_{24}, a_{42}, a_{35}, a_{53} > 0$, $a_{12}, a_{21}, a_{23}, a_{32}, a_{34}, a_{43}, a_{45}, a_{54} < 0$, $a_{14} = a_{41} = a_{25} = a_{52} = 0$, and diagonal entries are arbitrary. The graph $\Gamma(A)$ is given at Fig. 2; it satisfies the even-loop property. To construct the set V choose the vertex $v = 3$. Then $V = V_v^+ = \{1, 3, 5\}$, and $\sigma = (+, -, +, -, +)$ is the corresponding index of the graph $\Gamma(A)$. The field Ax has positive invariant orthants:

$$\mathbb{R}^5_\sigma = \{\, (x_1, \ldots, x_5) \in \mathbb{R}^5 \mid x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0,\ x_4 \leq 0,\ x_5 \geq 0 \,\}$$

and the opposite one

$$\mathbb{R}^5_{-\sigma} = \{\, (x_1, \ldots, x_5) \in \mathbb{R}^5 \mid x_1 \leq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ x_4 \geq 0,\ x_5 \leq 0 \,\}.$$

The graph $\Gamma(A)$ is connected, that is why there are no other positive invariant orthants.

The graph $\Gamma(-A)$ has odd loops (e.g., $(1, 2, 3)$), that is why the field Ax has no negative invariant orthants.

Remark. Example 1 shows that conditions of Lemma 4.1 [13] should be corrected. This lemma states (in terms of the current work) that the field Ax has positive invariant orthants iff the graph of the sign-symmetric matrix A has no odd loops *of length three*. For arbitrary sign-symmetric matrices A verification of loops of length three only is insufficient: in Example 1 the obstruction for existence of invariant orthants is a loop of length four. But it turns out that statement of Lemma 4.1 [13] is valid in generic case: if all off-diagonal entries of the matrix A are nonzero (see condition (4) below), then it is sufficient to verify loops of length three only. That is, the following proposition holds:

Theorem 5 *Let $A = (a_{ij})$ be a sign-symmetric $n \times n$ matrix with*

$$a_{ij} \neq 0 \quad \forall i \neq j. \tag{4}$$

The field Ax has positive (negative) invariant orthants iff all loops of length three of the graph $\Gamma(A)$ (resp. of the graph $\Gamma(-A)$) are even, or, which is equivalent,

$$a_{ij}a_{jk}a_{ki} > 0 \text{ (resp. } < 0) \quad \forall i \neq j \neq k \neq i. \tag{5}$$

Proof. Condition (4) implies that any two vertices in the graph $\Gamma(A)$ are connected by an edge. We show that if all loops of length three are even, then all loops of an arbitrary length are even. Choose any loop in the graph $\Gamma(A)$ and represent it as a sum of loops of length three. The sum of even loops is an even loop since negative edges of summands either annihilate in pairs and do not enter the sum (when they lie at adherent edges of the summands) or enter the sum (in the opposite case). Thus any loop in $\Gamma(A)$ is even. Now the statement of this theorem follows from Theorems 3, 4.

It is obvious that under condition (4) a loop of length three (i, j, k) is even if and only if $a_{ij}a_{jk}a_{ki} > 0$. For the graph $\Gamma(-A)$ this inequality turns into $(-a_{ij})(-a_{jk})(-a_{ki}) = -a_{ij}a_{jk}a_{ki} > 0$. □

Remark. Condition (4) may be changed by the weaker requirement that $a_{ij} \neq 0$ or $a_{ji} \neq 0$ for any $i \neq j$. Then the inequalities $a_{ij}a_{ji}a_{ik} > 0$ (resp. < 0), which characterize parity of the loop (i, j, k), should be changed by the inequalities $s(i, j)s(j, k)s(k, i) > 0$ (resp. < 0) with the use of the function $s(\cdot, \cdot)$ that determines sign of edges.

4 Invariant orthants of bilinear systems

Lemma 2 *If system (1) has a positive or negative invariant orthant, then the matrix A is sign-symmetric, and the matrices B_i, $i = 1, \ldots, m$, are diagonal.*

Proof. We show that an orthant positive (negative) invariant for system (1):

1. is positive (resp. negative) invariant for the field Ax;
2. both positive and negative invariant for any field $B_i x$, $i = 1, \ldots, m$.

Statement 1. For $u_1 = u_2 = \ldots = u_m \equiv 0$ trajectories of system (1) are trajectories of the field Ax.

Statement 2. For $u_i \neq 0$ and all the rest $u_1 = u_2 = \ldots = u_m \equiv 0$ we have $Ax + \sum_{j=1}^m u_j B_j x = Ax + u_i B_i x = (|u_i| Ax + \mathrm{sgn}\, u_i B_i x)/|u_i|$. A positive (negative) invariant orthant of the field $Ax + u_i Bx$ is positive (resp. negative) invariant for the field $|u_i| Ax + \mathrm{sgn}\, u_i Bx$. Passing to the limits $u_i \to +0$, $u_i \to -0$ and using continuous dependence of solution of a differential equation from the right-hand side, we obtain that the orthant under consideration is both positive and negative invariant for the field $B_i x$.

Statements 1, 2 are proved, and by virtue of Corollaries 1, 2, this lemma follows. □

Theorem 6 *Let $A, B_1, \ldots, B_m$ be $n \times n$ matrices. System (1) has positive (negative) invariant orthants iff the following conditions are satisfied:*

1. *the matrix A is sign-symmetric;*
2. *the matrices B_i, $i = 1, \ldots, m$, are diagonal;*
3. *the graph $\Gamma(A)$ (resp. $\Gamma(-A)$) satisfies the even-loop property.*

Then positive (negative) invariant orthants are $\mathbb{R}^n_\sigma$, where σ is any index of the graph $\Gamma(A)$ (resp. $\Gamma(-A)$), and their number is equal to 2^c, where c is the number of connected components of the graph $\Gamma(A)$.

Proof. Necessity. Items 1 and 2 follow from Lemma 2. Item 3 follows from Theorems 3, 4.

Sufficiency. All orthants in $\mathbb{R}^n$ are both positive and negative invariant for the fields $B_i x$ with the diagonal matrices B_i. Then the existence of invariant orthants and the formula for their number follow from Theorems 3, 4. □

Remarks. 1) If we assume in Theorem 6 that condition (4) holds, then by Theorem 5 we can restrict ourselves to verification of evenness of loops of length three only, i.e., of inequalities (5).

2) An index σ of the graph $\Gamma(A)$ is uniquely determined by a subset V of the set of vertices $\Gamma(A)$ that satisfies conditions a), b) of Theorem 1. Theorem 2 describes all such sets and gives a method of their construction. Thus we have a constructive method of enumeration of all invariant orthants.

5 Symmetric matrices and controllability

In this section we discuss the relation of our results with the following conjecture proposed by V. Jurdjevic and I. Kupka [6].

Conjecture. *If the matrices A and B are symmetric, then the single-input system*

$$\dot{x} = Ax + uBx, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u \in \mathbb{R}, \tag{6}$$

is not globally controllable in $\mathbb{R}^n \setminus \{0\}$.

Remark. There is an orthogonal transformation of $\mathbb{R}^n$ that diagonalizes a symmetric matrix B; then a symmetric matrix A turns into a symmetric one. That is why we can assume in the conjecture of V. Jurdjevic and I. Kupka that B is diagonal and A is symmetric.

Results of the previous section easily imply that this conjecture holds in dimensions 2 and 3: in fact, if A is sign-symmetric and B is diagonal, then system (6) has a positive or negative invariant orthant.

Even for $n = 4$ there are symmetric matrices A for which the field Ax and system (6) have no invariant orthants (see Example 1). Here the question of global controllability, i.e., of absence of *any* invariant sets, is left open. But for symmetric matrices A with at least one of the graphs $\Gamma(A)$, $\Gamma(-A)$ satisfying the even-loop property the conjecture of V. Jurdjevic and I. Kupka is now proved. However, in these cases not the symmetry but the sign-symmetry of A is essential.

Orthants are a very special kind of invariant domains for bilinear systems (6). But the following simple question seems to be open. Suppose that

$$B = \operatorname{diag}(b_1, \ldots, b_n), \quad b_i \neq b_j, \ i \neq j.$$

Is it true that if system (6) has no invariant orthants and everywhere satisfies the necessary Lie algebra rank controllability condition, then it has no invariant domains at all, i.e., it is globally controllable in $\mathbb{R}^n \setminus \{0\}$? In dimension 2 the answer is positive, but in greater dimensions it seems to be unknown.

One may lift system (1) to Lie groups $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{GL}_+(n, \mathbb{R})$, or homogeneous spaces of these groups, and study its global controllability on these state spaces (see [3, 6, 14] for this approach). It is well known that noncontrollability of a bilinear system on $\mathbb{R}^n \setminus \{0\}$ implies noncontrollability on $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{GL}_+(n, \mathbb{R})$ and their homogeneous spaces. Thus for matrices A, $B_1, \ldots, B_m$ satisfying conditions of Theorem 6 system (1) is not controllable on these Lie groups and their homogeneous spaces.

Acknowledgment.
The author wishes to thank Professor V. Jurdjevic for valuable comments and suggestions on this work. The author is also grateful to Dr. Yu.R. Romanovsky for bringing the results of the paper [4] to his attention.

References

[1] W.M. Boothby and E. Wilson, Determination of the transitivity of bilinear systems, *SIAM J. Control Optim.* **20** (1982), 634–644.

[2] C. Bruni, G. Di Pillo and G. Koch, Bilinear systems: an appealing class of "nearly linear" systems in theory and applications, *IEEE Trans. Autom. Control*, AC **19** (1974), 334–348.

[3] J.P. Gauthier and G. Bornard, Contrôlabilité des systèmes bilinèaires, *SIAM J. Control Optim.* **20** (1982), 3, 377–384.

[4] M.W. Hirsch, Convergence in neural nets, *Proceedings of the International Conference on Neural Networks*, vol. II, 1987, pp. 115–125, IEEE, USA.

[5] I. Joo, N.M. Tuan, On controllability of some bilinear systems, *Comptes Rendus de l'Academie des Sciences*, 1992, t. 315, Série I, 1393–1398.

[6] V. Jurdjevic and I. Kupka, Control Systems Subordinated to a Group Action: Accessibility, *J. Differ. Equat.* **39** (1981), 186–211.

[7] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[8] N.L. Lepe, Geometric method of investigation of controllability of two-dimensional bilinear systems, *Avtomatika i telemekhanika*, 1984, 11, 19–25 (translated into English in *Automatics and Remote Control*).

[9] R.R. Möhler, Bilinear control processes, in: *Mathematics in science and Engineering*, **106**, Academic Press, New York, 1973.

[10] R.E. Rink and R.R. Möhler, Completely controllable bilinear systems, *SIAM J. Control*, **6** (1968), 477–486.

[11] Yu.L. Sachkov, Controllability of three-dimensional bilinear systems, *Vestnik Moskovskogo Universiteta, ser. matem., mekh.*, 1991, No. 3, 26–30 (translated into English in *Moscow University Mathematics Bulletin*).

[12] Yu.L. Sachkov, Invariant domains of three-dimensional bilinear systems, *Vestnik Moskovskogo Universiteta, ser. matem., mekh.*, 1991, No. 4, 23–26 (translated into English in *Moscow University Mathematics Bulletin*).

[13] Yu.L. Sachkov, Invariant orthants of bilinear systems, *Differentcialnyje uravneniya*, 1995, No. 6, 1094–1095 (translated into English in *Differential Equations*).

[14] Yu.L. Sachkov, Controllability of Right-Invariant Systems on Solvable Lie Groups, *Journal of Dynamical and Control Systems* **3** (1997), No. 4, 531–564.