CONTROLLABILITY OF INVARIANT SYSTEMS ON LIE GROUPS AND HOMOGENEOUS SPACES

Yu. L. Sachkov UDC 517.977.1

Contents

1. Introduction

The aim of this work is to give a comprehensive survey of results on controllability of right-invariant control systems on Lie groups and their homogeneous spaces. This subject is an area of active research in the mathematical control theory and the Lie semigroup theory during the last 25 years. The motivations for this study are diverse: applications in mechanics and geometry, connections with other important classes of nonlinear control systems (bilinear and affine), the work on generalization of S. Lie's theory from the group case to the semigroup case.

The structure of this work is reflected in detail in the contents. First we give definitions, state the problems, and present general results on right-invariant systems (Secs. 2–4). In Secs. 5–7 we give controllability conditions in the three well-studied cases: homogeneous systems, compact Lie groups, and semidirect products. In the subsequent sections, we present controllability results according to group properties of the state space of a system: the semisimple case (Sec. 8), the nilpotent case (Sec. 9), and its generalization (Sec. 10), the solvable case and its generalizations (Secs. 11–16). In Sec. 17, we list some works related to the subject of this survey.

2. Definitions and General Properties of Right-Invariant Systems

Throughout this paper G will denote a real Lie group; L its Lie algebra, i.e., the set of right-invariant vector fields on G.

2.1. Basic definitions. A *right-invariant control system* Γ on a Lie group G is an arbitrary set of right-invariant vector fields on G, i.e., any subset

$$\Gamma \subset L. \tag{2.1}$$

A particular class of right-invariant systems, which is important for applications is formed by systems *affine in control*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}, \tag{2.2}$$

where A, B_1, …, B_m are some elements of L. If the control set U coincides with $\mathbb{R}^m$, then system (2.2) is an affine subspace of L.

Remark. Throughout this paper, we write a right-invariant control system as (2.1) or (2.2), i.e., as a set of vector fields, a *polysystem*. In the *classical notation*, control affine systems (2.2) are written as follows:

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x), \qquad u = (u_1, \dots, u_m) \in U, \quad x \in G, \tag{2.3}$$

with piecewise-constant control functions $u_1(\cdot)$, …, $u_m(\cdot)$. Polysystem (2.1) can also be written in such classical notation via a choice of a parametrization of the set Γ.

A *trajectory* of a right-invariant system Γ on G is a continuous curve $x(t)$ in G defined on an interval $[a, b] \subset \mathbb{R}$ so that there exists a partition $a = t_0 < t_1 < \dots < t_k = b$ and vector fields $A_1, \dots, A_k$ in Γ such that the restriction of $x(t)$ to each open interval (t_{i-1}, t_i) is differentiable and $\dot{x}(t) = A_i(x(t))$ for $t \in (t_{i-1}, t_i)$, $i = 1, \dots, k$.

For any $T \geq 0$ and any x in G, the *reachable set for time T* of a system Γ from the point x is the set $\mathbb{A}_\Gamma(x, T)$ of all points that can be reached from x in exactly T units of time:

$$\mathbb{A}_\Gamma(x, T) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x\}.$$

The *reachable set for time not greater than* $T \geq 0$ is defined as

$$\mathbb{A}_\Gamma(x, \leq T) = \bigcup_{0 \leq t \leq T} \mathbb{A}_\Gamma(x, t).$$

The *reachable* (or *attainable*) *set* of a system Γ from a point $x \in G$ is the set $\mathbb{A}_\Gamma(x)$ of all terminal points $x(T)$, $T \geq 0$, of all trajectories of Γ starting at x:

$$\mathbb{A}_\Gamma(x) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x,\ T \geq 0\} = \bigcup_{T \geq 0} \mathbb{A}_\Gamma(x, T).$$

If there is no ambiguity, in the sequel we denote the reachable sets $\mathbb{A}_\Gamma(x, T)$ and $\mathbb{A}_\Gamma(x)$ by $\mathbb{A}(x, T)$ and $\mathbb{A}(x)$, respectively.

A system Γ is called *controllable* if, given any pair of points x_0 and x_1 in G, the point x_1 can be reached from x_0 along a trajectory of Γ for a nonnegative time:

$$x_1 \in \mathbb{A}(x_0) \text{ for any } x_0,\ x_1 \in G,$$

or in other words, if

$$\mathbb{A}(x) = G \text{ for any } x \in G.$$

Another property, which is obviously weaker than controllability, is also essential for description of reachable sets. A system Γ is called *accessible* at a point $x \in G$ if the reachable set $\mathbb{A}(x)$ has nonempty interior in G.

The *orbit* of a system Γ passing through a point $x \in G$ is denoted by $\mathcal{O}_\Gamma(x)$ and is defined similar to the reachable set $\mathbb{A}(x)$, but the terminal time T can take both positive and negative values:

$$\mathcal{O}_\Gamma(x) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x,\ T \in \mathbb{R}\}.$$

If a system Γ is fixed, its orbit is denoted by $\mathcal{O}(x)$.

Remark. The inversion

$$i : G \to G, \qquad i(x) = x^{-1}$$

induces an isomorphism between the Lie algebra of right-invariant vector fields on a Lie group G and the Lie algebra of left-invariant vector fields on G. Thus, all problems for left-invariant control systems, including controllability, are reduced to the study of right-invariant systems.

For any subset $\Gamma \subset L$, we denote by $\operatorname{Lie}(\Gamma)$ the Lie algebra generated by Γ, i.e., the smallest subalgebra of L containing Γ.

Given any subset l of a vector space V, we denote by $\operatorname{span}(l)$ the vector subspace of V generated by l and by $\operatorname{co}(l)$ the positive convex cone generated by the set l.

We denote the topological closure and the interior of a set M by $\operatorname{cl} M$ and $\operatorname{int} M$, respectively.

The identity operator or matrix will be denoted by Id, and E_{ij} stands for the matrix with the identity ij-th entry and all other zero entries. We denote by A^{T} the transposed matrix of a matrix A.

2.2. Elementary properties of orbits and reachable sets. Let $\exp : L \to G$ be the exponential mapping from the Lie algebra L into the Lie group G. Any right-invariant field $A \in L$ is complete. The trajectory of A passing through the group identity $e \in G$ is

$$\exp(tA), \qquad t \in \mathbb{R},$$

and

$$\exp(tA)x, \qquad t \in \mathbb{R}$$

is the trajectory of A passing through a point $x \in G$.

Some properties of orbits of systems of right-invariant vector fields that are well known from the Lie group theory are collected in the following proposition.

Lemma 2.1. *Let $\Gamma \subset L$ be a right-invariant system, and let x be an arbitrary point of G. Then*

(i) $\mathcal{O}(x) = \{\exp(t_k A_k) \cdots \exp(t_1 A_1)x \mid A_i \in \Gamma,\ t_i \in \mathbb{R},\ k \in \mathbb{N}\}$;

(ii) $\mathcal{O}(x) = \mathcal{O}(e)x$;

(iii) $\mathcal{O}(e)$ *is the connected Lie subgroup of G with the Lie algebra* $\mathrm{Lie}(\Gamma)$;

(iv) $\mathcal{O}(x)$ *is the maximal integral manifold of the involutive right-invariant distribution* $\mathrm{Lie}(\Gamma)$ *on G passing through the point x.*

The following basic properties of attainable sets follow easily from the right-invariant property of Γ and the definition of $\mathbb{A}(x)$.

Lemma 2.2. *Let $\Gamma \subset L$ be a right-invariant system, and let x be an arbitrary point of G. Then*

(i) $\mathbb{A}(x) = \{\exp(t_k A_k) \cdots \exp(t_1 A_1)x \mid A_i \in \Gamma,\ t_i \geq 0,\ k \in \mathbb{N}\}$;

(ii) $\mathbb{A}(x) = \mathbb{A}(e)x$;

(iii) $\mathbb{A}(e)$ *is a subsemigroup of G*;

(iv) $\mathbb{A}(x)$ *is an arcwise-connected subset of G.*

Since all essential properties of attainable sets (including controllability, see, e.g., Theorems 2.6 and 2.7) are expressed in terms of the attainable set from the identity $\mathbb{A}(e)$, in the sequel, we restrict ourselves to this set and denote it by $\mathbb{A}$. In a similar way, we denote the orbit $\mathcal{O}(e)$ simply by $\mathcal{O}$.

2.3. Matrix systems. An important class of right-invariant systems that motivated the whole theory of such systems are *matrix* control systems.

Denote by $\mathrm{M}(n;\mathbb{R})$ the set of all $n \times n$ real matrices.

The *general linear group* $\mathrm{GL}(n;\mathbb{R})$ is formed by nonsingular real $n \times n$ matrices:

$$\mathrm{GL}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X \neq 0\}.$$

The group product in $\mathrm{GL}(n;\mathbb{R})$ is the usual matrix product, and the real analytic structure on $\mathrm{GL}(n;\mathbb{R})$ is induced by identifying $\mathrm{M}(n;\mathbb{R})$ with $\mathbb{R}^{n^2}$.

The Lie algebra of $\mathrm{GL}(n;\mathbb{R})$ is the space of all real $n \times n$ matrices:

$$\mathfrak{gl}(n;\mathbb{R}) = \mathrm{M}(n;\mathbb{R})$$

with the matrix commutator

$$[A, B] = AB - BA, \qquad A, B \in \mathfrak{gl}(n;\mathbb{R}),$$

as a Lie product.

Let G be a *linear group*, i.e., a closed subgroup of $\mathrm{GL}(n;\mathbb{R})$, and let $L \subset \mathfrak{gl}(n;\mathbb{R})$ be the Lie algebra of G.

For any matrix $A \in L$, the corresponding right-invariant vector field on G is defined by the matrix product

$$A(x) = Ax, \qquad x \in G \tag{2.4}$$

(we identify a right-invariant vector field with its value at the group identity).

The exponential mapping from L to G is the matrix exponential

$$A \mapsto \exp(A) = \mathrm{Id} + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots, \qquad A \in L.$$

The trajectory of $A \in L$ passing through a point $x \in G$ is given by the matrix exponential and the product

$$\exp(tA)x, \qquad t \in \mathbb{R}. \tag{2.5}$$

The right translation by an element $g \in G$

$$x \mapsto xg, \qquad x \in G$$

maps a trajectory (2.5) into a trajectory; this explains the name right-invariant for vector fields of the form (2.4).

A right-invariant control system on a linear group G is an arbitrary set of matrices $\Gamma \subset L$.

An affine in control right-invariant system on G has the form (2.2) for some matrices A, B_1, …, $B_m \in L$. In the classical notation, such system is written as a matrix control system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \qquad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in G. \tag{2.6}$$

Now we list several examples of linear groups G and their Lie algebras L. In each of these cases, G can be regarded as the state space of a right-invariant system $\Gamma \subset L$; in the affine in control case, see (2.2) or (2.6), the matrices A, B_1, …, B_m can arbitrarily be chosen in L.

Example 2.1. The general linear group $\mathrm{GL}(n;\mathbb{R})$ has the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$. Its dimension is equal to n^2. Notice that $\mathrm{GL}(n;\mathbb{R})$ is not connected: it has two connected components.

Example 2.2. The connected component of the identity in $\mathrm{GL}(n;\mathbb{R})$ is the group of all real $n \times n$ matrices with positive determinant:

$$\mathrm{GL}_+(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X > 0\}.$$

The Lie algebra of the group $\mathrm{GL}_+(n;\mathbb{R})$ is $\mathfrak{gl}(n;\mathbb{R})$.

Example 2.3. The *special linear group* is the group of all real $n \times n$ unimodular matrices:

$$\mathrm{SL}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X = 1\}.$$

It is a connected (n^2-1)-dimensional Lie group, and its Lie algebra $\mathfrak{sl}(n;\mathbb{R})$ consists of all $n \times n$ matrices with zero trace:

$$\mathfrak{sl}(n;\mathbb{R}) = \{A \in \mathrm{M}(n;\mathbb{R}) \mid \mathrm{tr} A = 0\}.$$

Example 2.4. The *special orthogonal group* is formed by all real $n \times n$ orthogonal unimodular matrices:

$$\mathrm{SO}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid X^{\mathrm{T}} = X^{-1},\ \det X = 1\}.$$

It is a connected Lie group of dimension $n(n-1)/2$, and its Lie algebra $\mathfrak{so}(n;\mathbb{R})$ consists of all real $n \times n$ skew-symmetric matrices:

$$\mathfrak{so}(n;\mathbb{R}) = \{A \in \mathrm{M}(n;\mathbb{R}) \mid A^{\mathrm{T}} = -A\}.$$

2.4. Normal accessibility. If a point y in G is reachable (or accessible) from a point x in G, then there exist elements $A_1, \ldots, A_k$ in Γ and $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ with positive coordinates such that

$$y = \exp(t_k A_k) \cdots \exp(t_1 A_1) x.$$

The following stronger notion turns out to be important in the study of topological properties of reachable sets and controllability.

Definition 2.1. A point y in G is called *normally accessible* from a point x in G by Γ if there exist elements $A_1, \ldots, A_k$ in Γ and $\hat{t} \in \mathbb{R}^k$ with positive coordinates $\hat{t}_1, \ldots, \hat{t}_k$ such that the mapping $F(t_1, \ldots, t_k) = \exp(t_k A_k) \cdots \exp(t_1 A_1) x$ as a mapping from $\mathbb{R}^k$ into G satisfies the following conditions:

(i) $F(\hat{t}) = y$.

(ii) The rank of the differential dF at $\hat{t}$ is equal to the dimension of G.

The point y is said to be normally accessible from x by $A_1, \ldots, A_k$.

Theorem 2.1. *If* $\operatorname{Lie}(\Gamma) = L$*, then in any neighborhood* O *of the identity* $e \in G$*, there are points normally accessible from* e *by* Γ*. Consequently, the set* $\operatorname{int} \mathbb{A} \cap O$ *is nonempty.*

Proof. Denote $n = \dim L = \dim \operatorname{Lie}(\Gamma)$. If $n = 0$, everything is clear. Assume that $n > 0$ and fix a neighborhood O of the identity e.

There exists a nonzero element $A_1 \in \Gamma$. The curve

$$M_1 = \{\exp(t_1 A_1) \mid 0 < t_1 < \varepsilon_1\}$$

is a smooth one-dimensional manifold contained in the neighborhood O for a sufficiently small positive ε_1. If $n = 1$, then any point in M_1 is normally accessible from e by A_1, since the mapping $F_1(t_1) = \exp(t_1 A_1)$ has rank 1 on the interval $I_1 = (0, \varepsilon_1)$.

If $n > 1$, there exists an element $A_2 \in \Gamma$ such that the right-invariant field A_2 is not tangent to M_1 at any point of M_1; if this is the case for any $A_2 \in \Gamma$, then $\dim \operatorname{Lie}(\Gamma) = 1$; a contradiction. That is why the set

$$M_2 = \{\exp(t_2 A_2) \exp(t_1 A_1) \mid 0 < t_i < \varepsilon_i,\ i = 1, 2\}$$

is a smooth two-dimensional manifold that belongs to O for sufficiently small positive ε_1 and ε_2. Moreover, the mapping $F_2(t_1, t_2) = \exp(t_2 A_2) \exp(t_1 A_1)$ has rank 2 in the domain $I_2 = (0, \varepsilon_1) \times (0, \varepsilon_2)$. If $n = 2$, the theorem is proved, since in this case, any point of M_2 is normally accessible from e by A_1 and A_2.

If $n > 2$, we proceed by induction. For any dimension $k < n$ and some elements $A_1, \ldots, A_k \in \Gamma$, we construct the k-dimensional smooth manifold

$$M_k = \{\exp(t_k A_k) \cdots \exp(t_1 A_1) \mid 0 < t_i < \varepsilon_i,\ i = 1, \ldots, k\}$$

contained in the neighborhood O for sufficiently small positive $\varepsilon_1, \ldots, \varepsilon_k$, so that the mapping $F_k(t_1, \ldots, t_k) = \exp(t_k A_k) \cdots \exp(t_1 A_1)$ has the rank k in the domain $I_k = (0, \varepsilon_1) \times \cdots \times (0, \varepsilon_k)$. Then any point in M_n is normally accessible from e by $A_1, \ldots, A_n$.

The image of the box I_n by the mapping F_n is an open set contained in $\mathbb{A}$ and O; thus, $\operatorname{int} \mathbb{A} \cap O \supset F_n(I_n)$.

If the Lie algebra generated by Γ does not coincide with the whole Lie algebra L, then Γ can be considered as a right-invariant system on the orbit $\mathcal{O}$. By item (iv) of Lemma 2.1, $\operatorname{Lie}(\Gamma)$ coincides with the Lie algebra of the Lie group $\mathcal{O}$; thus, the previous theorem implies the following relationship between the attainable set $\mathbb{A}$ and the orbit $\mathcal{O}$.

Lemma 2.3.

(i) *The attainable set* $\mathbb{A}$ *is contained in the orbit* $\mathcal{O}$.

(ii) *For any neighborhood* O *of the identity* e *in the topology of the orbit* $\mathcal{O}$*, the intersection* $\operatorname{int}_{\mathcal{O}} \mathbb{A} \cap O$ *is nonempty.*

(iii) *Moreover,* $\operatorname{cl}\operatorname{int}_{\mathcal{O}} \mathbb{A} \supset \mathbb{A}$.

(We denote by $\operatorname{int}_{\mathcal{O}}$ the interior of a subset of the orbit $\mathcal{O}$ in the topology of $\mathcal{O}$.)

Proof. Item (i) is straightforward. Item (ii) follows from Theorem 2.1: since $\operatorname{Lie}(\Gamma)$ is the Lie algebra of $\mathcal{O}$, one should replace in this theorem G by $\mathcal{O}$. To prove inclusion (iii), take any point x in $\mathbb{A}$ and choose any neighborhood U of x in $\mathcal{O}$. We have to show that the intersection $\operatorname{int}_{\mathcal{O}} \mathbb{A} \cap U$ is nonempty. There exists a neighborhood O of e in $\mathcal{O}$ such that $Ox \subset U$. By item (ii), there is a point y in $\operatorname{int}_{\mathcal{O}} \mathbb{A} \cap O$. Then $yx \in \operatorname{int}_{\mathcal{O}} \mathbb{A} \cap U$.

2.5. Basic controllability conditions.

Theorem 2.2. *A necessary condition for a right-invariant system Γ on G to be controllable is that the Lie group G be connected.*

Proof. The reachable set $\mathbb{A}$ is arcwise-connected; see Lemma 2.2.

Remark. In view of the previous theorem, in the sequel, *all Lie groups are assumed to be connected*, unless otherwise explicitly specified.

The fundamental necessary controllability condition given in the following proposition is usually referred to as the *rank condition* or the *Lie algebra rank condition* (*LARC*).

Theorem 2.3. *A necessary condition for a right-invariant system Γ on G to be controllable is that Γ generates L as a Lie algebra*: $\operatorname{Lie}(\Gamma) = L$. *If* $\operatorname{Lie}(\Gamma) = L$, *then the attainable set $\mathbb{A}$ has a nonempty interior in the group G.*

Proof. If $\mathbb{A} = G$, then more so $\mathcal{O} = G$. By Lemma 2.1, $\operatorname{Lie}(\Gamma) = L$.

If $\operatorname{Lie}(\Gamma) = L$, then Theorem 2.1 yields $\operatorname{int} \mathbb{A} \neq \varnothing$.

In general, the rank condition is not sufficient for controllability, but it is equivalent to accessibility.

Theorem 2.4. *A right-invariant system Γ on G is accessible at the identity* (*and thus at any point in G*) *if and only if* $\operatorname{Lie}(\Gamma) = L$.

Proof. *Necessity.* If the reachable set $\mathbb{A}$ has a nonempty interior in G, then the same holds for the orbit $\mathcal{O}$. By Lemma 2.1, we obtain $\operatorname{Lie}(\Gamma) = L$.

Sufficiency. If $\operatorname{Lie}(\Gamma) = L$, then $\operatorname{int} \mathbb{A}$ is nonempty by Theorem 2.1.

A system $\Gamma \subset L$ is said to *have a full rank* if the rank condition $\operatorname{Lie}(\Gamma) = L$ holds.

Theorem 2.5. *A right-invariant system Γ on a connected Lie group G is controllable if and only if the following conditions hold*:

(i) *The attainable set $\mathbb{A}$ is a subgroup of G and*

(ii) $\operatorname{Lie}(\Gamma) = L$.

Proof. *Necessity.* Item (i) is obvious, and item (ii) follows from the rank condition.

Sufficiency. If $\mathbb{A}$ is a subgroup, then for any exponential $\exp(tA)$, $A \in \Gamma$, $t \geq 0$, its inverse $\exp(-tA)$ is also in $\mathbb{A}$. Thus, the attainable set A coincides with the orbit $\mathcal{O}$. But since Γ has the full rank, its orbit coincides with the whole group G (see Lemma 2.1, item (iv)). Consequently, $\mathbb{A} = G$.

Theorem 2.6. *A right-invariant system Γ is controllable on a connected Lie group G if and only if it is controllable from the identity, i.e., $\mathbb{A} = G$.*

Proof. Apply item (ii) of Lemma 2.2.

For a full-rank system Γ, its attainable set $\mathbb{A}$ has a nonempty interior in G. But in general, the identity e can lie on the boundary of $\mathbb{A}$.

Theorem 2.7. *A right-invariant system Γ is controllable on a connected Lie group G if and only if the group identity e is contained in the interior of $\mathbb{A}$.*

Proof. Necessity is obvious, and sufficiency follows from the fact that for a connected Lie group G, an arbitrary neighborhood of the identity e generates G as a semigroup.

The following controllability condition is fundamental, since it shows us that in the study of controllability of full-rank systems, one can replace the attainable set $\mathbb{A}$ by its closure $\operatorname{cl} \mathbb{A}$.

Theorem 2.8. *If the reachable set $\mathbb{A}$ is dense in a connected Lie group G and* $\operatorname{Lie}(\Gamma) = L$*, then Γ is controllable on G.*

Proof. Consider the backward-time system

$$-\Gamma = \{-A \mid A \in \Gamma\};$$

its trajectories are trajectories of Γ passed in the backward time. The attainable set of $-\Gamma$ is

$$\mathbb{A}_{-\Gamma} = \{\exp(-t_k A_k) \cdots \exp(-t_1 A_1) \mid A_i \in \Gamma,\ t_i \geq 0,\ k \in \mathbb{N}\} = \mathbb{A}^{-1}. \qquad (2.7)$$

Since the system $-\Gamma$ has the full rank: $\operatorname{Lie}(-\Gamma) = \operatorname{Lie}(\Gamma) = L$, its attainable set has a nonempty interior and thus contains an open set O_1.

On the other hand, since Γ has the full rank, there is a point x in G that has a neighborhood $O(x)$ contained in $\mathbb{A}$.

The closure of the attainable set from x is everywhere dense: $\operatorname{cl} \mathbb{A}(x) = \operatorname{cl}(\mathbb{A} \cdot x) = G$; thus, there exists a point $y \in \mathbb{A}(x) \cap O_1$. We have $y \in \mathbb{A} \cdot x$; hence $yx^{-1} \in \mathbb{A}$. Taking into account the inclusion $O(x) \subset \mathbb{A}$ and the semigroup property of $\mathbb{A}$, we obtain that the neighborhood $O(y) = yx^{-1} \cdot O(x)$ of the point y is contained in $\mathbb{A}$. But $y \in O_1 \subset \mathbb{A}^{-1}$; thus, $y^{-1} \in \mathbb{A}$, and the neighborhood of the identity $O(e) = y^{-1} \cdot O(y)$ is contained in $\mathbb{A}$. By Theorem 2.7, $\mathbb{A} = G$.

Theorem 2.9. *A right-invariant system Γ is controllable on a connected Lie group G if and only if the identity e is normally accessible from e by some elements $A_1, \ldots, A_l$ in Γ.*

Proof. *Necessity.* By Theorem 2.1, there exists a point $x \in G$ that is normally accessible by some fields $A_1, \dots, A_k \in \Gamma$ from e. Since Γ is controllable, the backward-time system $-\Gamma$ is also controllable; thus,

$$e = \exp(t_l A_l) \cdots \exp(t_{k+1} A_{k+1}) x$$

for some $A_{k+1}, \dots, A_l \in \Gamma$ and some $t_{k+1}, \dots, t_l > 0$. Then e is normally accessible from e by the fields $A_1, \dots, A_l$.

Sufficiency follows from Theorem 2.7, since a normally accessible point is in the interior of the attainable set.

The preceding result easily implies that controllability of right-invariant systems is preserved under small perturbations. More precisely, let $\rho(\cdot, \cdot)$ be the distance in the Lie algebra L, and let $d(\cdot, \cdot)$ be the corresponding Hausdorff distance between subsets of L:

$$d(\Gamma_1, \Gamma_2) = \max \left\{ \sup_{A_1 \in \Gamma_1} \inf_{A_2 \in \Gamma_2} \rho(A_1, A_2), \sup_{A_2 \in \Gamma_2} \inf_{A_1 \in \Gamma_1} \rho(A_1, A_2) \right\}.$$

Theorem 2.10. *If a right-invariant system $\Gamma \subset L$ is controllable, then there exists $\varepsilon > 0$ such that any system $\Gamma' \subset L$ is controllable provided that $d(\Gamma, \Gamma') < \varepsilon$.*

Proof. If Γ is controllable, then the identity e is normally accessible from e by some $A_1, \dots, A_k \in \Gamma$. For a sufficiently small $\varepsilon > 0$, any system Γ' with $d(\Gamma, \Gamma') < \varepsilon$ contains elements $A'_1, \dots, A'_k$ such that $\rho(A_i, A'_i) < \varepsilon$, $i = 1, \dots, k$. Then e is normally accessible from e by $A'_1, \dots, A'_k$.

2.6. Remarks. Control systems with a Lie group as a state space are studied in the mathematical control theory since the early 1970-ies.

Brockett [36] considered applied problems leading to control systems on matrix groups and their homogeneous spaces; e.g., a model of DC to DC conversion and the rigid body control raise control problems on the group of rotations of the three-space $\mathrm{SO}(3; \mathbb{R})$ and on $\mathrm{SO}(3; \mathbb{R}) \times \mathbb{R}^3$, respectively. The natural framework for such problems are matrix control systems of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) B_i x(t), \quad u_i(t) \in \mathbb{R}, \tag{2.8}$$

where $x(t)$ and A, $B_1, \dots, B_m$ are $n \times n$ matrices.

The systematic mathematical study of control systems on Lie groups was initiated by Jurdjevic and Sussmann [83]. They observed that the passage from the matrix system (2.8) to a more general right-invariant system

$$\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} u_i(t) B_i(x(t)), \quad x(t) \in G, \quad u(t) \in \mathbb{R},$$

where A, $B_1, \dots, B_m$ are right-invariant vector fields on a Lie group G, “in no essential way affects the nature of the problem”. The basic properties of attainable set and orbits of right-invariant systems were found in [83].

The notion of normal accessibility (for arbitrary nonlinear systems) is due to Sussmann [140].

3. Control Systems Subordinated to a Group Action

3.1. Transitive actions, homogeneous spaces, and controllability.

Definition 3.1. A Lie group G is said to act on an analytic manifold M if there exists an analytic mapping $\theta : G \times M \to M$ that satisfies the following conditions:

(1) $\theta(g_2 g_1, x) = \theta(g_2, \theta(g_1, x))$ for any g_1, g_2 in G and any x in M;

(2) $\theta(e, x) = x$ for any x in M.

For each $g \in G$, consider the analytic diffeomorphism $\theta_g : M \to M$ given by $\theta_g(x) = \theta(g, x)$ (the inverse to θ_g is given by $\theta_{g^{-1}}$). The mapping $g \mapsto \theta_g$ is called an *action* of G on M. Any action is a homomorphism from the group G into the group of analytic diffeomorphisms of M. For any element $A \in L$, $\theta_{\exp tA}$ is a one-parameter group of diffeomorphisms of M with the generator $\theta_*(A)$, an analytic vector field on M:

$$\theta_*(A)(x) = \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp tA}(x), \quad x \in M, \ A \in L.$$

Such vector fields $\theta_*(A)$, $A \in L$ are said to be *subordinated to the action θ of G*. They form the finite-dimensional Lie algebra

$$\theta_*(L) = \{\theta_*(A) \mid A \in L\}$$

of complete vector fields on M.

Definition 3.2. A system of vector fields $\mathcal{F}$ on M is called *subordinated to an action θ* if $\mathcal{F}$ is contained in $\theta_*(L)$. If $\mathcal{F} = \theta_*(\Gamma)$ for some right-invariant system $\Gamma \subset L$, then $\mathcal{F}$ is said to be *induced* by Γ.

A Lie group G acts *transitively* on M if, for any $x \in M$, the orbit $\{\theta_g(x) \mid g \in G\}$ coincides with the whole M. A manifold that admits a transitive action of a Lie group is called the *homogeneous space* of this Lie group. Homogeneous spaces are exactly manifolds that can be represented as quotients of Lie groups. If θ is a transitive action of G on M, then we can consider the isotropy group H at a given point $x \in M$:

$$H = \{g \in G \mid \theta_g(x) = x\}.$$

H is a closed subgroup of G, and the manifold M is diffeomorphic to the left coset space G/H with the diffeomorphism $G/H \to M$ given by $gH \mapsto \theta_g(x)$.

Given a right-invariant system Γ on a Lie group G that acts on a manifold M, one can construct a system on M induced by Γ. The following proposition is a controllability result related to this construction.

Theorem 3.1. *Let θ be an action of a connected Lie group on a manifold M, $\Gamma \subset L$ be a right-invariant system on G, and let $\mathcal{F} = \theta_*(\Gamma)$ be the induced system on M.*

(i) *For any point x in M, the reachable set of $\mathcal{F}$ from x is*

$$\mathbb{A}_{\mathcal{F}}(x) = \theta_{\mathbb{A}_\Gamma}(x) = \{\theta_g(x) \mid g \in \mathbb{A}_\Gamma\}.$$

(ii) *Assume that the action θ is transitive. If Γ is controllable on G, then $\mathcal{F}$ is controllable on M.*

(iii) *$\mathcal{F}$ is controllable on M if and only if the semigroup $\mathbb{A}_\Gamma$ acts transitively on M.*

Proof. (i) For any trajectory $g(t)$ of Γ and for any point x in M, the curve $\theta_{g(t)}(x)$ is a trajectory of $\mathcal{F}$; moreover, any trajectory of $\mathcal{F}$ is obtained in such a way.

(ii) If $\mathbb{A}_\Gamma = G$, then $\mathbb{A}_{\mathcal{F}}(x) = M$, since the orbit of θ coincides with M.

(iii) Sufficiency follows in the same way as in (ii). The necessity is obtained from the description of the reachable set $\mathbb{A}_{\mathcal{F}}(x)$ in (i).

Important applications of Theorem 3.1 are related to the linear action of linear groups $G \subset \mathrm{GL}(n;\mathbb{R})$ on the vector space $\mathbb{R}^n$. In this case, the induced systems are bilinear, or more generally, affine systems.

3.2. Bilinear systems.

3.2.1. Induced vector fields and systems. For the linear action of the group $\mathrm{GL}(n;\mathbb{R})$ on the vector space $\mathbb{R}^n$,

$$\theta_g(x) = gx, \quad g \in \mathrm{GL}(n;\mathbb{R}),\ x \in \mathbb{R}^n,$$

the induced vector fields are linear:

$$\theta_*(A)(x) = Ax, \quad A \in \mathfrak{gl}(n;\mathbb{R}),\ x \in \mathbb{R}^n.$$

Given any elements $A, B_1, \dots, B_m \in \mathfrak{gl}(n;\mathbb{R})$ and a control set $U \subset \mathbb{R}^m$, consider the following right-invariant system on $\mathrm{GL}(n;\mathbb{R})$, which is affine in control:

$$\Gamma = \left\{ A + \sum_{i=1}^m u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}.$$

Then the induced system is the following set of linear vector fields on $\mathbb{R}^n$:

$$\mathcal{F} = \left\{ A + \sum_{i=1}^m u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}.$$

Passing from polysystems to control systems in the classical notation, we obtain a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^m u_i B_i x, \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n.$$

3.2.2. Bilinear systems on $\mathbb{R}^n \setminus \{0\}$. Assume that the action of a connected linear group $G \subset \mathrm{GL}(n;\mathbb{R})$ is transitive on the punctured vector space $M =$

$\mathbb{R}^n \setminus \{0\}$. The typical examples are the groups $\mathrm{GL}_+(n;\mathbb{R})$ and $\mathrm{SL}(n;\mathbb{R})$. Let L be the Lie algebra of G. The Lie algebras in the previous examples are respectively $\mathfrak{gl}(n;\mathbb{R})$ and $\mathfrak{sl}(n;\mathbb{R})$.

For this case, Theorem 3.1 implies the following.

Corollary 3.1. *If a right-invariant system*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

is controllable on a linear group G that acts transitively on $\mathbb{R}^n \setminus \{0\}$, then the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n \setminus \{0\}$$

is controllable on $\mathbb{R}^n \setminus \{0\}$.

3.2.3. Bilinear systems on S^{n-1}. Now consider the case of a connected linear group whose action is transitive on the unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\},$$

e.g., the group $\mathrm{SO}(n;\mathbb{R})$ of rotations of $\mathbb{R}^n$. Let L be the Lie algebra of G. In the previous example, the Lie algebra $\mathfrak{so}(n;\mathbb{R})$ is formed by $n \times n$ skew-symmetric matrices.

Then Theorem 3.1 yields the following.

Corollary 3.2. *If a right-invariant system*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

is controllable on a linear group G that acts transitively on S^{n-1}, then the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \quad x \in S^{n-1}$$

is controllable on the sphere S^{n-1}.

3.3. Affine systems.

3.3.1. Induced vector fields and systems. Let $\mathrm{Aff}(n;\mathbb{R})$ be the group of invertible affine transformations of $\mathbb{R}^n$. It is the semidirect product of the group of translations of $\mathbb{R}^n$ with the general linear group:

$$\mathrm{Aff}(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{GL}(n;\mathbb{R}).$$

This group can be represented as a subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ by matrices of the form

$$\overline{X} = \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix}, \quad X \in \mathrm{GL}(n;\mathbb{R}),\ x \in \mathbb{R}^n.$$

Embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the hyperplane

$$\mathbb{R}^n \times \{v_{n+1} = 1\} = \{(v_1, \ldots, v_n, 1)^{\mathrm{T}} \in \mathbb{R}^{n+1} \mid (v_1, \ldots, v_n)^{\mathrm{T}} \in \mathbb{R}^n\},$$

we obtain an affine mapping in $\mathbb{R}^n$ defined by an element $\overline{X} \in \mathrm{Aff}(n;\mathbb{R})$; this is the mapping

$$\begin{pmatrix} v \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Xv + x \\ 1 \end{pmatrix}.$$

That is, the group $\mathrm{Aff}(n;\mathbb{R})$ acts on $\mathbb{R}^n$ as follows:

$$\theta_{\overline{X}}(v) = Xv + x, \quad \overline{X} \in \mathrm{Aff}(n;\mathbb{R}), \quad v \in \mathbb{R}^n.$$

The Lie algebra $\mathfrak{aff}(n;\mathbb{R})$ of the affine group is represented by the matrices

$$\overline{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \quad A \in \mathfrak{gl}(n;\mathbb{R}),\ a \in \mathbb{R}^n.$$

The one-parameter subgroup in $\mathrm{Aff}(n;\mathbb{R})$ corresponding to $\overline{A} \in \mathfrak{aff}(n;\mathbb{R})$ is

$$\exp t \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{tA} & \frac{e^{tA} - \mathrm{Id}}{A} a \\ 0 & 1 \end{pmatrix},$$

where

$$\frac{e^{tA} - \mathrm{Id}}{A} = t \,\mathrm{Id} + \frac{t^2}{2!} A + \cdots + \frac{t^n}{n!} A^{n-1} + \cdots .$$

The corresponding flow in $\mathbb{R}^n$ is

$$\theta_{\exp(t\overline{A})}(v) = e^{tA} v + \frac{e^{tA} - \mathrm{Id}}{A} a;$$

thus, the induced vector field is an affine field on $\mathbb{R}^n$:

$$\theta_*(\overline{A})(v) = Av + a, \quad v \in \mathbb{R}^n.$$

Now let G be a connected linear subgroup of $\mathrm{Aff}(n;\mathbb{R})$ that acts transitively on $\mathbb{R}^n$, e.g., the group of invertible affine transformations of $\mathbb{R}^n$ that preserve the orientation

$$\mathrm{Aff}_+(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{GL}_+(n;\mathbb{R}) = \left\{ \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \mid X \in \mathrm{GL}_+(n;\mathbb{R}),\ x \in \mathbb{R}^n \right\},$$

or the group of Euclidean motions of $\mathbb{R}^n$,

$$\mathrm{E}(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{SO}(n;\mathbb{R}) = \left\{ \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \mid X \in \mathrm{SO}(n;\mathbb{R}),\ x \in \mathbb{R}^n \right\}.$$

Let L be the Lie algebra of G; in the previous cases, we have the Lie algebras

$$\mathfrak{aff}(n;\mathbb{R}) = \mathbb{R}^n \rtimes \mathfrak{gl}(n;\mathbb{R}) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(n;\mathbb{R}),\ a \in \mathbb{R}^n \right\},$$

and

$$\mathfrak{e}(n;\mathbb{R}) = \mathbb{R}^n \rtimes \mathfrak{so}(n;\mathbb{R}) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{so}(n;\mathbb{R}),\ a \in \mathbb{R}^n \right\},$$

respectively. A right-invariant system

$$\Gamma = \left\{ \overline{A} + \sum_{i=1}^{m} u_i \overline{B}_i \mid u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L \tag{3.1}$$

on the Lie group G that is affine in control with

$$\overline{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \quad \overline{B}_i = \begin{pmatrix} B_i & b_i \\ 0 & 0 \end{pmatrix}, \ i = 1, \ldots, m,$$

induces the following *affine* control system:

$$\dot{x} = Ax + a + \sum_{i=1}^{m} u_i (B_i x + b_i), \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Notice that particular cases of affine systems are bilinear systems considered in Secs. 3.2.2 and 3.2.3 ($a = b_1 = \cdots = b_m = 0$) and the classical *linear* systems

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i b_i, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n,$$

obtained in the case $a = 0$, $B_1 = \cdots = B_m = 0$.

Now Theorem 3.1 implies the following proposition.

Corollary 3.3. *Let G be a connected linear subgroup of* $\mathrm{Aff}(n;\mathbb{R})$ *that acts transitively on $\mathbb{R}^n$. If a right-invariant system* (3.1) *is controllable on G, then the induced affine system* (3.2) *is controllable on $\mathbb{R}^n$.*

3.4. Remarks. Control systems on homogeneous spaces that are subordinated to a group action (in particular, bilinear and affine systems) were among the most important motivations for the study of right-invariant systems. The contents of this section is mainly due to Brockett [36]. The terminology used and the general approach were adopted by Jurdjevic and Kupka [80].

Boothby and Wilson [29, 31] found a complete list of linear groups that act transitively on $\mathbb{R}^n \setminus \{0\}$. Moreover, they presented an algorithm for verification whether a Lie group generated by given matrices belongs to this list; this algorithm involves only rational matrix operations.

Lie groups that act transitively on spheres are also listed; see Samelson [128], p. 26, Borel [32, 33], Montgomery and Samelson [105].

4. Lie Saturate

An efficient method for obtaining (sufficient) controllability conditions for right-invariant systems is the extension technique based on the computation of the tangent cone to the closure of the attainable set of a system at the group identity.

Definition 4.1. Two right-invariant systems $\Gamma_1, \Gamma_2 \subset L$ are called *equivalent* one another if $\mathrm{cl}(\mathbb{A}_{\Gamma_1}) = \mathrm{cl}(\mathbb{A}_{\Gamma_2})$.

Definition 4.2. Let $\Gamma \subset L$ be a right-invariant system. The Lie saturate of Γ, denoted by $\mathrm{LS}(\Gamma)$, is the largest subset of $\mathrm{Lie}(\Gamma)$ that is equivalent to Γ.

If two systems Γ_1 and Γ_2 are equivalent to Γ, then their union $\Gamma_1 \cup \Gamma_2$ is obviously equivalent to Γ. That is why the Lie saturate of Γ always exists: it is the union of all systems in $\mathrm{Lie}(\Gamma)$ that are equivalent to Γ. The largest right-invariant system that is equivalent to Γ is $\{A \in L \mid \exp(tA) \in \mathrm{cl}(\mathbb{A}_\Gamma) \ \forall t \geq 0\}$; thus, the Lie saturate can be described as follows.

Theorem 4.1. *For any system* $\Gamma \subset L$,

$$\mathrm{LS}(\Gamma) = \mathrm{Lie}(\Gamma) \cap \{A \in L \mid \exp(tA) \in \mathrm{cl}(\mathbb{A}_\Gamma) \ \forall t \geq 0\}.$$

Denote by $\mathrm{E}(\Gamma)$ the set $\{A \in \mathrm{LS}(\Gamma) \mid -A \in \mathrm{LS}(\Gamma)\}$. It is the largest vector subspace of L contained in $\mathrm{LS}(\Gamma)$.

The basic properties of Lie saturate are collected in the following proposition.

Theorem 4.2.

(0) $\mathrm{LS} \circ \mathrm{LS} = \mathrm{LS}$;

(1) $\mathrm{LS}(\Gamma)$ *is a closed convex positive cone in* L*, i.e.,*

(1a) $\mathrm{LS}(\Gamma)$ *is topologically closed*:

$$\mathrm{cl}(\mathrm{LS}(\Gamma)) = \mathrm{LS}(\Gamma),$$

(1b) $\mathrm{LS}(\Gamma)$ *is convex*:

$$A, B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A + (1-\alpha)B \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha \in [0,1],$$

(1c) $\mathrm{LS}(\Gamma)$ *is a positive cone*:

$$A \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha \geq 0.$$

Thus,

$$A, B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A + \beta B \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha, \beta \geq 0.$$

(2) *For any* $A \in \mathrm{E}(\Gamma)$ *and for any* $t \in \mathbb{R}$,

$$e^{t \operatorname{ad} A} \operatorname{LS}(\Gamma) \subset \operatorname{LS}(\Gamma).$$

That is,

$$\pm A, B \in \operatorname{LS}(\Gamma) \;\Rightarrow\; e^{t \operatorname{ad} A} B = B + (t \operatorname{ad} A) B + \frac{(t \operatorname{ad} A)^2}{2!} B + \ldots \in \operatorname{LS}(\Gamma) \quad \forall\, t \in \mathbb{R}.$$

(3) $\mathrm{E}(\Gamma)$ *is a subalgebra of* L. *In particular,*

$$\pm A, \pm B \in \operatorname{LS}(\Gamma) \;\Rightarrow\; \pm [A, B] \in \operatorname{LS}(\Gamma).$$

(4) *If* $A \in \operatorname{LS}(\Gamma)$ *and if the one-parameter subgroup* $\{\exp(tA) \mid t \in \mathbb{R}\}$ *is periodic, then* $\mathbb{R}A \subset \operatorname{LS}(\Gamma)$.

Proof. (0) is obvious in view of the definition of the Lie saturate and Theorem 4.1.

(1) follows from the well-known properties $\mathbb{A}_{\operatorname{cl}(\Gamma)} \subset \operatorname{cl}(\mathbb{A}_\Gamma)$, $\mathbb{A}_{\operatorname{co}(\Gamma)} \subset \operatorname{cl}(\mathbb{A}_\Gamma)$, and $\mathbb{A}_{\mathbb{R}_+\Gamma} = \mathbb{A}_\Gamma$ of reachable sets.

To prove (2), assume that $\pm A, B \in \operatorname{LS}(\Gamma)$. Then

$$\exp(s e^{t \operatorname{ad} A} B) = \exp(s \operatorname{Ad}_{\exp(tA)} B) = \exp(tA) \exp(sB) \exp(-tA) \in \operatorname{cl}(\mathbb{A}_\Gamma)$$

for any $s \geq 0$, $t \in \mathbb{R}$; thus $e^{t \operatorname{ad} A} B \in \operatorname{LS}(\Gamma)$ for all $t \in \mathbb{R}$.

Now (3) easily follows: if $\pm A, \pm B \in \operatorname{LS}(\Gamma)$, then $\pm e^{t \operatorname{ad} A} B, \pm B \in \operatorname{LS}(\Gamma)$, that is why

$$\pm [A, B] = \pm \lim_{t \to 0} \frac{e^{t \operatorname{ad} A} B - B}{t} \in \operatorname{LS}(\Gamma).$$

(4) follows from the chain

$$\{\exp(tA) \mid t \geq 0\} = \{\exp(tA) \mid t \in \mathbb{R}\} \subset \mathbb{A}_\Gamma,$$

which is valid for all $A \in \operatorname{LS}(\Gamma)$ with a periodic one-parameter group.

The following theorem gives a general controllability test in terms of the Lie saturate.

Theorem 4.3. *A right-invariant system* $\Gamma \subset L$ *is controllable on a connected Lie group* G *if and only if* $\operatorname{LS}(\Gamma) = L$.

Proof. *Necessity* follows from the definition of the Lie saturate.

Sufficiency. Assume that $\operatorname{LS}(\Gamma) = L$. The connected Lie group G is generated by the one-parameter semigroups $\{\exp(tA) \mid A \in L,\ t \geq 0\}$ as a semigroup; thus, $\operatorname{cl}(\mathbb{A}) = G$. If, in addition, the rank condition $\operatorname{Lie}(\Gamma) = L$ holds, then Γ is controllable by Theorem 2.8.

Usually, it is difficult to construct the Lie saturate of a right-invariant system explicitly. That is why Theorem 4.3 is applied as a sufficient condition of controllability via the following procedure. Starting from a given system Γ, one constructs a completely ordered ascending family of extensions $\{\Gamma_\alpha\}$ of Γ, i.e.,

$$\Gamma_0 = \Gamma, \qquad \Gamma_\alpha \subset \Gamma_\beta \text{ if } \alpha < \beta.$$

The extension rules are provided by Theorem 4.2:

(1) given Γ_α, one constructs $\Gamma_\beta = \operatorname{cl}(\operatorname{co}(\Gamma_\alpha))$;

(2) for $\pm A, B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup e^{\mathbb{R}\operatorname{ad} A}B$;

(3) for $\pm A, \pm B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}[A, B]$;

(4) given $A \in \Gamma_\alpha$ with periodic one-parameter group, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}A$.

Theorem 4.2 guarantees that all extensions Γ_α belong to $\operatorname{LS}(\Gamma)$. If one obtains the relation $\Gamma_\alpha = L$ at some step α, then $\operatorname{LS}(\Gamma) = L$, and the system Γ is controllable.

4.1. Remarks. The idea to consider the closure of attainable sets as an invariant of right-invariant systems is important in controllability questions and goes back to Jurdjevic and Sussmann [83]. The concept of Lie saturate and the extension technique were developed by Jurdjevic and Kupka [80, 81].

The Lie subsemigroup theory studies general subsemigroups of Lie groups, not necessarily appearing as reachable sets of right-invariant systems. A generalization of Theorem 4.2 holds for this case.

A subset W of a Lie algebra L is called a *wedge* if W is a closed positive convex cone in L. The *edge* of a wedge W, denoted by $H(W)$, is the maximal vector subspace of L contained in W:

$$H(W) = W \cap -W.$$

A wedge W is called a *Lie wedge* if

$$e^{\operatorname{ad} A} W \subset W \quad \text{for all } A \in H(W).$$

For a closed subsemigroup S of a Lie group G that contains the identity element e, its tangent object

$$L(S) = \{A \in L \mid \exp(tA) \in S \ \forall t \geq 0\}$$

is the Lie wedge.

The basic results on the subsemigroup theory can be found in books by Hofmann and Lawson [66], Hilgert and Neeb [59], and Hilgert, Hofmann and Lawson [58].

5. Homogeneous Systems

5.1. Controllability criterion. A system $\Gamma \subset L$ is called *homogeneous* if, together with any element X, this system contains also the sign-opposite element $-X$, i.e.,

$$\Gamma = -\Gamma.$$

Theorem 5.1. *Let Γ be a homogeneous right-invariant system on G. Then its reachable set $\mathbb{A}$ is a subgroup of G and coincides with the orbit $\mathcal{O}$.*

Proof. Apply Lemmas 2.1 and 2.2.

Thus the study of controllability for Γ is reduced to the verification of the algebraic condition of coincidence of the connected Lie groups $\mathcal{O}$ and G.

Theorem 5.2. *A homogeneous right-invariant system $\Gamma \subset L$ is controllable on a connected Lie group if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. By Lemma 2.1, the Lie algebra of the Lie group $\mathcal{O}$ is $\mathrm{Lie}(\Gamma)$. Then apply Theorem 5.1.

5.2. Control-affine systems. A control-affine system

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}$$

is homogeneous if the drift term A is equal to zero and the control set U is symmetric with respect to the origin: $U = -U$. For this case, Theorems 5.1 and 5.2 are specified as follows.

Theorem 5.3. *Assume that a control set $U \subset \mathbb{R}^m$ satisfies the relation $U = -U$. Consider the homogeneous control-affine system*

$$\Gamma = \left\{ \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

on a Lie group G. Then

(i) *The reachable set $\mathbb{A}$ coincides with the orbit $\mathcal{O}$, i.e., with the connected Lie subgroup of G with the Lie algebra* $\mathrm{Lie}(\Gamma)$;

(ii) *If $U = \mathbb{R}^m$, then any point of $\mathbb{A}$ can be reached from the identity e at an arbitrary time*:
$$\mathbb{A}(e, T) = \mathbb{A} = \mathcal{O} \textit{ for any } T > 0;$$

(iii) *If G is connected and $U = \mathbb{R}^m$, then the system Γ is controllable if and only if* $\mathrm{Lie}(B_1, \dots, B_m) = L$.

Proof. Items (i) and (iii) follow respectively from Theorems 5.1 and 5.2.

To prove (ii), choose any $T > 0$. Let a point x in G be reachable from e for some time $T_1 > 0$:

$$x = \exp(t_k X_k) \cdots \exp(t_1 X_1), \quad \sum_{i=1}^{k} t_i = T_1,$$

where $t_1, \dots, t_k > 0$ and $X_1, \dots, X_k \in \Gamma$. Since the control set $U = \mathbb{R}^m$ is homothetic with respect to the origin, the vector fields $Y_i = \alpha X_i$, $i = 1, \dots, k$, belong to Γ for $\alpha = T_1/T > 0$. Thus, x can be reached from e for time T:

$$x = \exp(s_k Y_k) \cdots \exp(s_1 Y_1), \quad \sum_{i=1}^{k} s_i = T,$$

where $s_i = t_i/\alpha$, $i = 1, \dots, k$.

5.3. Remarks. The controllability criterion for homogeneous matrix systems was given by Brockett [36]. In this paper, the criterion was also specified for the group of matrices $\mathrm{GL}_+(n;\mathbb{R})$ with positive determinant, the group of unimodular matrices $\mathrm{SL}(n;\mathbb{R})$, the group of symplectic matrices $\mathrm{Sp}(n;\mathbb{R})$, and the group of orthogonal unimodular matrices $\mathrm{SO}(n;\mathbb{R})$.

The general controllability results for homogeneous right-invariant systems on Lie groups are due to Jurdjevic and Sussmann [83].

6. Compact Lie Groups

In this section, we consider the case of a Lie group that is *compact* as a topological space.

6.1. Controllability conditions.

Theorem 6.1. *A right-invariant system $\Gamma \subset L$ is controllable on a compact connected Lie group G if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. For any right-invariant vector field $A \in L$ on a compact Lie group G, the negative and positive semitrajectories satisfy the inclusion

$$\mathrm{cl}\{\exp(-tA) \mid t \geq 0\} \subset \mathrm{cl}\{\exp(tA) \mid t \geq 0\}.$$

That is why any right-invariant system Γ on G is equivalent to the homogeneous system $\Gamma \cup -\Gamma$. But for homogeneous systems, controllability is equivalent to the rank condition; see Theorem 5.2.

Theorem 6.2. *Let a Lie group G be compact and connected, and let a right-invariant system $\Gamma \subset L$ be controllable on G. Then there exists $T > 0$ such that for every $g_0, g_1 \in G$, there is a control that steers g_0 to g_1 for not more than T units of time.*

Proof. The interiors of the reachable sets $\mathbb{A}(e, \leq t)$, $t \geq 0$, form an open covering of the group G. By compactness of G, there is an instant $T_1 > 0$ such that

$$\operatorname{int} \mathbb{A}(e, \leq T_1) = G.$$

That is, the identity element e can be steered to any element $g_1 \in G$ for not more than T_1 units of time. A similar argument applied to $-\Gamma$ shows us that there exists $T_2 > 0$ such that any element $g_0 \in G$ can be steered to e for not more than T_2 units of time. Then g_0 and g_1 can be connected by a trajectory of Γ for time not more than $T = T_1 + T_2$.

6.2. Examples.

6.2.1. Special orthogonal group in dimension 3. Let $G = \mathrm{SO}(3; \mathbb{R})$, the set of all 3×3 real orthogonal matrices with positive determinant. The Lie group G is compact and connected. Its Lie algebra $L = \mathfrak{so}(3; \mathbb{R})$ is the set of all 3×3 real skew-symmetric matrices.

Take any linearly independent matrices $A_1, A_2 \in \mathfrak{so}(3; \mathbb{R})$ and consider the right-invariant system $\Gamma = \{A_1, A_2\}$. Notice that the matrices A_1, A_2, and $[A_1, A_2]$ span the whole Lie algebra $\mathfrak{so}(3; \mathbb{R})$. By Theorem 6.1, the system Γ is controllable. That is, any rotation in $\mathrm{SO}(3; \mathbb{R})$ can be written as the product of exponentials

$$\exp(t_k A_{i_k}) \cdots \exp(t_1 A_{i_1}), \quad t_j \geq 0, \; i_j \in \{1, 2\}, \; k \in \mathbb{N}. \tag{6.1}$$

Moreover, by Theorem 6.2, there is $T > 0$ that gives a universal upper bound $\sum_{j=1}^{k} t_j \leq T$ for decomposition (6.1) of any rotation in $\mathrm{SO}(3; \mathbb{R})$.

The single-input right-invariant affine in control system

$$\dot{X} = (A_1 + uA_2)X, \quad u \in U \subset \mathbb{R}, \; X \in \mathrm{SO}(3; \mathbb{R}) \tag{6.2}$$

is also controllable (for any control set U containing more than one element). Moreover, there is $T > 0$ such that given any two matrices $P, Q \in \mathrm{SO}(3; \mathbb{R})$, there is a piecewise-constant control that steers P to Q for not more than T units of time. Notice that in general, there may not exist arbitrarily small numbers T with the above property even if the control is unconstrained, i.e., $U = \mathbb{R}$. Take, for instance, $A_1 = E_{12} - E_{21}$ and $A_2 = E_{13} - E_{31}$. Write the solution to system (6.2) with the initial condition $X(0) = \mathrm{Id}$ as $X = (x_{ij})_{i,j=1,2,3}$. Then we have

$$\begin{aligned} \dot{x}_{12} &= x_{22} + ux_{32}, \\ \dot{x}_{32} &= -ux_{12}. \end{aligned}$$

Multiplying the first equation by x_{12}, the second equation by x_{32}, and adding, we obtain

$$\frac{1}{2} \frac{d}{dt}(x_{12}^2 + x_{32}^2) = x_{22} x_{12}.$$

Since $x_{12}^2 + x_{32}^2$ vanishes at $t = 0$, we have

$$(x_{12}^2 + x_{32}^2)(t) = 2\int_0^t x_{22}(\tau)x_{12}(\tau)\,d\tau.$$

But $x_{22}(\tau)$ and $x_{12}(\tau)$ are entries of the orthogonal matrix $X(\tau)$; hence, their absolute values are bounded by 1. Therefore, we conclude that

$$(x_{12}^2 + x_{32}^2)(t) \le 2t.$$

This shows us that a matrix (a_{ij}) for which $a_{12}^2 + a_{32}^2 = 1$ cannot be reached from the identity for less than $\frac{1}{2}$ units of time.

6.2.2. Special orthogonal group in dimension n. The previous considerations are generalized to the group $G = \mathrm{SO}(n;\mathbb{R})$ of rotations of $\mathbb{R}^n$. In this case, the Lie algebra L of G is the set of all $n \times n$ skew-symmetric matrices $\mathfrak{so}(n;\mathbb{R})$.

Take the matrices $A_1 = \sum_{i=1}^{n-2}(E_{i,i+1} - E_{i+1,i})$ and $A_2 = E_{n-1,n} - E_{n,n-1}$. It is easy to show that $\mathrm{Lie}(A_1, A_2) = \mathfrak{so}(n;\mathbb{R})$. Thus, even though the group $\mathrm{SO}(n;\mathbb{R})$ is $\frac{1}{2}n(n-1)$-dimensional, the system

$$\dot{X} = (A_1 + uA_2)X, \quad X \in \mathrm{SO}(n;\mathbb{R}),\ u \in U \subset \mathbb{R},$$

in which only one control is involved, is controllable (if the control set U contains at least two distinct points).

Moreover, as above, we can find an upper bound for time that is necessary for reaching one point in $\mathrm{SO}(n;\mathbb{R})$ from another.

Notice that the set of pairs (A_1, A_2) such that $\mathrm{Lie}(A_1, A_2) = L$ is open and dense in $L \times L$ (this is valid for any semisimple Lie algebra L; see Theorem 8.1 below). Thus, we can replace the matrices A_1 and A_2 by an "almost arbitrary" pair in $L \times L$.

6.2.3. Serret–Frenet frames. Let $x(t)$ denote any curve in a Euclidean space $\mathbb{R}^n$ whose derivatives $d^k x(t)/dt^k$, $k = 1, \ldots, n$, span an n-dimensional vector space at each point along the curve. The Serret–Frenet frame along the curve x is described by an orthonormal matrix $R(t)$ in $\mathrm{SO}(n;\mathbb{R})$ that relates this frame to a standard orthonormal frame $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ and that further satisfies the following differential equation in $\mathrm{SO}(n;\mathbb{R})$:

$$\frac{dR}{dt} = R(t)\begin{pmatrix} 0 & -k_1(t) & 0 & \ldots & 0 \\ k_1(t) & 0 & -k_2(t) & & \vdots \\ 0 & k_2(t) & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{n-1}(t) \\ 0 & \ldots & 0 & k_{n-1}(t) & 0 \end{pmatrix}, \qquad (6.3)$$

where $k_1(t), \ldots, k_{n-1}(t)$ are called the curvature functions associated with the curve x. (For curves in $\mathbb{R}^3$, k_2 is called the torsion of x.) Notice that the curvatures

$k_1, \dots, k_{n-2}$ are positive, while the last curvature k_{n-1} could be of any sign. The curve

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and the rotation matrix $R(t)$ can be expressed as the curve

$$g(t) = \begin{pmatrix} 1 & 0 \\ x(t) & R(t) \end{pmatrix}$$

in the group $\mathrm{E}(n;\mathbb{R})$ of motions of $\mathbb{R}^n$ realized as the closed subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ consisting of all matrices

$$\begin{pmatrix} 1 & 0 \\ x & R \end{pmatrix}, \qquad x \in \mathbb{R}^n,\ R \in \mathrm{SO}(n;\mathbb{R}).$$

Since the first vector in the Serret–Frenet frame coincides with the tangent vector dx/dt, it follows that $dx/dt = R(t)e_1$, where $e_1 = (1, 0, \dots, 0)^{\mathrm{T}}$. Being combined with system (6.3) for the orientation matrix $R(t)$, this gives the following left-invariant control affine system in $\mathrm{E}(n;\mathbb{R})$:

$$\frac{dg}{dt} = g(t) \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -k_1(t) & 0 & \cdots & 0 \\ 0 & k_1(t) & 0 & -k_2(t) & & \vdots \\ 0 & 0 & k_2(t) & 0 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & -k_{n-1}(t) \\ 0 & 0 & \cdots & 0 & k_{n-1}(t) & 0 \end{pmatrix}, \tag{6.4}$$

with $k_1, \dots, k_{n-1}$ playing the role of controls.

Consider the extreme case where all, except for one, curvatures are constant. Then Eq. (6.3) can be written as the control affine system

$$\frac{dR}{dt} = R(t)(A + uB), \quad R \in \mathrm{SO}(n;\mathbb{R}),\ u \geq 0, \tag{6.5}$$

where $u(t) = k_i(t)$ is the nonconstant curvature (we assume that $1 \leq i \leq n-2$; in the case $i = n-1$, the control should be unconstrained: $u \in \mathbb{R}$), and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{6.6}$$

$$A_1 = \begin{pmatrix} 0 & -k_1 & 0 & \cdots & 0 \\ k_1 & 0 & -k_2 & & \vdots \\ 0 & k_2 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{i-1} \\ 0 & \cdots & 0 & k_{i-1} & 0 \end{pmatrix}, \tag{6.7}$$

$$A_2 = \begin{pmatrix} 0 & -k_{i+1} & 0 & \cdots & 0 \\ k_{i+1} & 0 & -k_{i+2} & & \vdots \\ 0 & k_{i+2} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{n-1} \\ 0 & \cdots & 0 & k_{n-1} & 0 \end{pmatrix}, \tag{6.8}$$

and

$$B = \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix} = E_{i+1,i} - E_{i,i+1}. \tag{6.9}$$

Writing $h(t)$ for $R^{-1}(t)$ turns the left-invariant system (6.5) into the right-invariant system

$$\frac{dh}{dt} = -(A + uB)h(t), \qquad h \in \mathrm{SO}(n; \mathbb{R}), \; u \geq 0, \tag{6.10}$$

which will be called the *Serret–Frenet system*. It follows from Theorem 6.1 that system (6.10) is controllable if and only if the set $\Gamma = \{-A - uB \mid u \geq 0\}$ generates $\mathfrak{so}(n; \mathbb{R})$ as a Lie algebra, i.e., $\mathrm{Lie}(A, B) = \mathfrak{so}(n; \mathbb{R})$. A description of the Lie algebra $\mathrm{Lie}(A, B)$ is given in the following proposition.

Theorem 6.3. *Assume that each fixed curvature k_j, $j \neq i$, in Eqs.* (6.7) *and* (6.8) *is nonzero. The Lie algebra generated by the matrices A and B, which is given by* (6.6)–(6.9)*, is equal to $\mathfrak{so}(n; \mathbb{R})$ in all the cases, except for one. The exceptional case occurs when $n = 2m$, $i = m$, and $k_1 = \cdots = k_{m-1} = k_{m+1} = \cdots = k_{n-1}$. The Lie algebra in the exceptional case is equal to the Lie algebra of the unitary group $\mathrm{U}(2m; \mathbb{R})$.*

6.3. Homogeneous spaces.

6.3.1. Sphere. The $(n-1)$-sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is the homogeneous space of the group $\mathrm{SO}(n; \mathbb{R})$ of rotations of $\mathbb{R}^n$.

Let A, B_1, ..., B_m be $n \times n$ skew-symmetric matrices. The control-affine right-invariant system

$$\dot{X} = (A + \sum_{i=1}^{m} u_i B_i)X, \quad X \in \mathrm{SO}(n; \mathbb{R}), \; u = (u_1, \ldots, u_m) \in \mathbb{R}^m, \tag{6.11}$$

induces the bilinear system

$$\dot{x} = (A + \sum_{i=1}^{m} u_i B_i)x, \quad x \in S^{n-1}, \; u = (u_1, \ldots, u_m) \in \mathbb{R}^m; \tag{6.12}$$

one can consider the unit vector $x \in S^{n-1}$ to be the first column of the orthogonal matrix $X \in \mathrm{SO}(n; \mathbb{R})$. Theorems 6.1 and 6.2 imply the following proposition.

Corollary 6.1. *Let matrices $A, B_1, \ldots, B_m \in \mathfrak{so}(n; \mathbb{R})$ generate $\mathfrak{so}(n; \mathbb{R})$ as a Lie algebra. Then system* (6.12) *is globally controllable on the sphere S^{n-1}. Moreover, there exists $T > 0$ such that any points $x_0, x_1 \in S^{n-1}$ can be connected by a trajectory of* (6.12) *corresponding to a piecewise constant control for not more than T units of time.*

Remark. System (6.12) is globally controllable on the sphere S^{n-1} if and only if the reachable set $\mathbb{A}$ of system (6.11), which is always a subgroup of $\mathrm{SO}(n; \mathbb{R})$, acts transitively on S^{n-1}.

Each group $\mathrm{SO}(n; \mathbb{R})$ and $\mathrm{U}(2m; \mathbb{R})$, $2m = n$, acts linearly on $\mathbb{R}^n$ by the left multiplication, and these actions are transitive on the spheres in $\mathbb{R}^n$. That is why Theorems 6.1 and 6.3 yield the following.

Corollary 6.2. *Let matrices A and B be given by* (6.6)–(6.9)*. If all curvatures k_j, $j \neq i$, are nonzero, then the bilinear system*

$$\dot{x} = Ax + uBx, \qquad x \in S^{n-1}, \; u \geq 0$$

is controllable on the sphere S^{n-1}.

6.3.2. Grassmann manifolds. The Grassmann manifold $G(k, n)$ consists of all k-dimensional vector subspaces of $\mathbb{R}^n$. The manifold structure on $G(k, n)$ can be introduced by embedding it into the *orthogonal group*

$$\mathrm{O}(n; \mathbb{R}) = \{X \in \mathrm{M}(n; \mathbb{R}) \mid X^{\mathrm{T}} = X^{-1}\}.$$

Each k-dimensional subspace $S \in G(k, n)$ can be identified with the orthogonal reflection $P_S \in \mathrm{O}(n; \mathbb{R})$ given by $P_S(x) = x$ for $x \in S$ and $P_S(x) = -x$ for x in the orthogonal complement of S. The requirement that the correspondence $S \mapsto P_S$ is a homeomorphism turns $G(k, n)$ into a topological space. Since $\{P_S \mid S \in G(k, n)\}$ is a closed subset of the compact Lie group $\mathrm{O}(n; \mathbb{R})$, then $G(k, n)$ is compact.

The group $\mathrm{O}(n; \mathbb{R})$ acts on $G(k, n)$ in a natural way: for any $S \in G(k, n)$ and any $R \in \mathrm{O}(n; \mathbb{R})$, the subspace $RS = \{Rx \mid x \in S\}$ is an element of $G(k, n)$. This action is transitive. In terms of the correspondence $S \mapsto P_S$, it is expressed as $RS \mapsto RP_S R^{\mathrm{T}}$ with R^{T} being equal to the transpose of R. The isotropy group is $H = \mathrm{O}(n - k; \mathbb{R}) \times \mathrm{O}(k; \mathbb{R})$; thus $\dim G(k, n) = \dim \mathrm{O}(n; \mathbb{R}) - \dim H = k(n - k)$.

For any skew-symmetric matrix A and each S in $G(k, n)$,

$$\left.\frac{d}{dt}\right|_{t=0} (\exp tA) P_S (\exp -tA) = AP_S - P_S A = [A, P_S].$$

That is, $X_A(S) = [A, P_S]$ is an infinitesimal generator of the one-parameter group of isomorphisms induced by A.

Let $\mathcal{P}$ denote the set of all vector fields X_A in $G(k,n)$ with A in $\mathfrak{so}(n;\mathbb{R})$. $\mathcal{P}$ is the family of vector fields in $G(k,n)$ subordinated to the group action of $\mathrm{O}(n;\mathbb{R})$ on $G(k,n)$. A description of the reachable set $\mathbb{A}_{\mathcal{F}}$ of any subfamily $\mathcal{F} \subset \mathcal{P}$ is obtained from Theorems 3.1 and 6.1.

Theorem 6.4.

(a) *The reachable set $\mathbb{A}_{\mathcal{F}}(x)$ of $\mathcal{F}$ from any point $x \in G(k,n)$ is equal to the orbit of $\mathcal{F}$ passing through x.*

(b) *Let Γ denote the set of all matrices A such that X_A is in $\mathcal{F}$. Then $\mathbb{A}_{\mathcal{F}}(x) = Gx = \{gxg^{\mathrm{T}} \mid g \in G\}$ with G being equal to $\mathbb{A}_\Gamma$, i.e., the subgroup of* $\mathrm{SO}(n;\mathbb{R})$ *generated by $\{\exp tA \mid A \in \Gamma, t \in \mathbb{R}\}$.*

(c) *$\mathcal{F}$ is controllable on $G(k,n)$ if and only if G acts transitively on $G(k,n)$.*

6.4. Remarks. The controllability results of Sec. 6.1 and their application to the group of rotations in Secs. 6.2.1 and 6.2.2 are due to Jurdjevic and Sussmann [83].

Serret–Frenet frames (Sec. 6.2.3) were studied by Jurdjevic [79]. The proof of Theorem 6.3 on the Lie algebra generated in the control problem on $\mathrm{SO}(n;\mathbb{R})$ with one fixed curvature can be found in [74].

The applications to Grassmann manifolds (Sec. 6.3.2) is also due to Jurdjevic [79].

By the argument of Sec. 6.2.1, Theorems 6.1 and 6.2 can be regarded as results that describe the generation of compact Lie groups. Related results on the generation of both compact and noncompact classical Lie groups can be found in Sec. 8.6 and in papers by Crouch and Silva Leite [43], Silva Leite [132, 133, 134, 135], and Albuquerque and Silva Leite [5].

7. Semidirect Products of Lie Groups

In this section, we consider the case of a Lie group G that is a semidirect product of a vector space V by a Lie group K. If K is compact, then complete controllability results are available; in particular, if the Lie group K has no nonzero fixed points in the space V, then the rank condition is equivalent to the controllability. If K is not compact, then controllability conditions are obtained by considering compact subgroups of K.

Let K and V be Lie groups, and let K act on V. Consider the *semidirect product* $G = V \ltimes K$. The manifold G is the Cartesian product of V and K, and the group product in G is defined by

$$(v_1, k_1) \cdot (v_2, k_2) = (v_1 + k_1 v_2, k_1 k_2), \quad v_1, v_2 \in V,\ k_1, k_2 \in K.$$

The Lie algebra L of G is the *semidirect sum* $L(V) \rtimes L(K)$, where $L(V)$ and $L(K)$ are Lie algebras of V and K, respectively. The vector space L is the direct sum of the vector spaces $L(V)$ and $L(K)$, and the Lie algebra product in L is as follows:

$$[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2]+b_1(a_2)-b_2(a_1), [b_1, b_2]), \;\; a_1, a_2 \in L(V), \; b_1, b_2 \in L(K).$$

Denote the projections from G onto the factors V and K by τ and π, respectively,

$$\tau \,:\, G \to V, \quad \tau(v, k) = v, \qquad v \in V, \; k \in K,$$
$$\pi \,:\, G \to K, \quad \pi(v, k) = k, \qquad v \in V, \; k \in K.$$

The projection π is a Lie group homomorphism. Denote by $L(K)$ the Lie algebra of K. The differential

$$\pi_* \,:\, L \to L(K)$$

is a Lie algebra homomorphism.

Throughout this section, we assume that V is a vector Lie group, i.e., a finite-dimensional real vector space regarded as an Abelian Lie group. In addition, we assume that the action of the Lie group K on the vector space V is linear.

Definition 7.1. We say that $v \in V$ is a *fixed point* under K if

$$Kv = \{gv \mid g \in K\} = \{v\}.$$

We write this as $Kv = v$.

Notice that the origin $0 \in V$ is a fixed point for any linear action on V.

7.1. K is compact and admits no nonzero fixed points in V. In this subsection, we prove the following result, which can be considered as a generalization of the controllability test for compact Lie groups (Theorem 6.1).

Theorem 7.1. *Let a compact connected Lie group K act linearly on a vector space V, and let V admit no nonzero fixed points with respect to K. Then a right-invariant system $\Gamma \subset L$ is controllable on the Lie group $G = V \ltimes K$ if and only if* $\mathrm{Lie}(\Gamma) = L$.

7.1.1. Proof of Theorem 7.1 in particular cases. Before proving the theorem in its full generality, we give a shorter proof for the most interesting in applications cases $G = \mathrm{E}(n; \mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{SO}(n; \mathbb{R})$ and $G = \mathbb{R}^{2m} \ltimes \mathrm{U}(2m; \mathbb{R})$.

Proof. The rank condition $\mathrm{Lie}(\Gamma) = L$ is necessary for controllability of Γ by Theorem 2.3.

Assume that $\mathrm{Lie}(\Gamma) = L$. Then the right-invariant system $\Gamma_K = \pi_*(\Gamma)$ on K is controllable, since K is compact and connected; see Theorem 6.1. That is,

$$\pi(\mathbb{A}) = K. \tag{7.1}$$

It follows from Theorem 2.7 that it is sufficient to show that the identity $e = (0, \mathrm{Id}) \in G$ is contained in the interior of $\mathbb{A}$. Let (x, k) be a point in the interior of $\mathbb{A}$, which is nonempty by the rank condition. In view of (7.1), there exists $y \in V$

such that (y, k^{-1}) is contained in $\mathbb{A}$. Then $(x, k)(y, k^{-1}) = (x + ky, \mathrm{Id})$, and this product is in the interior of $\mathbb{A}$.

Denote $x + ky$ by v. Let Ω be a neighborhood of Id in K such that $(v, \Omega) \subset \operatorname{int} \mathbb{A}$.

For any $h \in \Omega$ and $n \in \mathbb{N}$, the element $(v, h)^n = (v + hv + \cdots + h^{n-1}v, h^n)$ is contained in the interior of $\mathbb{A}$. If $h^n = \mathrm{Id}$ and if $v = hw - w$ for some $w \in V$, then $v + hv + \cdots + h^{n-1}v = 0$, and $e = (0, \mathrm{Id})$ is contained in the interior of $\mathbb{A}$. To complete the proof, we have to show in either of the two cases ($K = \mathrm{SO}(n; \mathbb{R})$, $V = \mathbb{R}^n$, and $K = \mathrm{U}(2m; \mathbb{R})$, $V = \mathbb{R}^{2m}$) that for any $v \in V$ and any neighborhood Ω of Id in K, there exists an element h in Ω such that $v \in \operatorname{Im}(h - \mathrm{Id})$ and $h^m = \mathrm{Id}$ for some $m \in \mathbb{N}$.

We outline a proof for the first case; for the second one, it is similar. Let P be a plane in $\mathbb{R}^n$, $n \geq 2$, that contains a given point $v \in \mathbb{R}^n$. Then, for any neighborhood Ω of Id in the group of rotations of the plane P, there exists a rotation $R \in \Omega$ such that $R - \mathrm{Id}$ is nonsingular and $R^m = \mathrm{Id}$ for some $m \in \mathbb{N}$. Then R can be extended to $\mathbb{R}^n$ by setting it equal to the identity on the orthogonal complement of P in $\mathbb{R}^n$. Hence $v \in \operatorname{Im}(R - \mathrm{Id})$ and $R^m = \mathrm{Id}$.

7.1.2. Proof of Theorem 7.1 in the general case.

We first obtain several auxiliary propositions under the condition $\mathrm{Lie}(\Gamma) = L$.

Lemma 7.1. $\pi(\mathbb{A}) = K$.

Proof. The projected system $\Gamma_K = \pi_*(\Gamma)$ is a full-rank right-invariant system on the compact connected Lie group K; hence, it is controllable on K; see Theorem 6.1.

In the next three lemmas, we study the following subset of G:

$$T = \{(v, \mathrm{Id}) \mid (v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}\}. \tag{7.2}$$

Lemma 7.2. *T is nonempty.*

Proof. By Theorem 2.4, the system Γ is accessible, i.e., the interior of $\mathbb{A}$ is nonempty. Take any $(w, g) \in \operatorname{int} \mathbb{A}$. By virtue of Lemma 7.1, there exists $v \in V$ such that $(v, g^{-1}) \in \mathbb{A}$. Then

$$(w, g) \cdot (v, g^{-1}) = (w + gv, \mathrm{Id}) \in \operatorname{int} \mathbb{A}.$$

Hence T is nonempty.

Lemma 7.3. *For each $(v, \mathrm{Id}) \in T$, there exists an integer $N > 0$ such that $(\lambda v, \mathrm{Id}) \in T$ for all λ with $\lambda > N$.*

Proof. If $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$, then there exist $\varepsilon > 0$ such that $((1+\lambda)v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for all λ with $|\lambda| < \varepsilon$. Hence $((1+\lambda)v, \mathrm{Id})^n = (n(1+\lambda)v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for each integer $n > 0$. Let N be any integer with $N(1+\varepsilon) > 1$. Then the closed real interval $[N, N+1]$ has the property that $(\lambda v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for all $\lambda \in [N, N+1]$.

But, by the semigroup property of $\mathbb{A}$, the whole real ray $\{\lambda \mid \lambda > N\}$ has such a property.

Lemma 7.4. *For each* $(v, \mathrm{Id}) \in T$ *and for each* $g \in K$*, there exists an integer* $M > 0$ *such that* $(Mgv, \mathrm{Id}) \in T$*.*

Proof. For each $g \in K$, by Lemma 7.1, there exist vectors $v_g, v_{g^{-1}} \in V$ such that (v_g, g) and $(v_{g^{-1}}, g^{-1})$ belong to $\mathbb{A}$. Hence $(v_{g^{-1}}, g^{-1}) \cdot (v_g, g) = (g^{-1}v_g + v_{g^{-1}}, \mathrm{Id})$ belongs to $\mathbb{A}$.

If $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$, then let $M > 0$ be any integer such that

$$\left(v - M^{-1}\left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right)$$

belongs to $\operatorname{int} \mathbb{A}$. Therefore,

$$\left(v - M^{-1}\left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right)^M = \left(Mv - \left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right) \in \operatorname{int} \mathbb{A}.$$

But then

$$(v_g, g) \cdot \left(Mv - \left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right) \cdot \left(v_{g^{-1}}, g^{-1}\right) = (Mgv, \mathrm{Id})$$

belongs to $\operatorname{int} \mathbb{A}$.

Now we prove Theorem 7.1.

Proof. The rank condition $\mathrm{Lie}(\Gamma) = L$ is necessary for controllability by Theorem 2.3. In order to prove the sufficiency, assume that $\mathrm{Lie}(\Gamma) = L$.

By Lemma 7.2, there exists a vector $v \in V$ such that $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$. Let

$$\bar{v} = \int_K Kv \, d\mu,$$

where μ is a Haar measure on K such that $\mu(K) = 1$. Then $K\bar{v} = \bar{v}$, and by the hypothesis of the theorem, $\bar{v} = 0$.

On the other hand, the mean $\int_K Kv \, d\mu$ is contained in the convex cone generated by the set $\{gv \mid g \in K\}$, that is why

$$0 = \bar{v} = \sum_{j=1}^{p} \lambda_j g_j v \quad \text{for some } g_1, \dots, g_p \in K, \quad \lambda_1 > 0, \dots, \lambda_p > 0.$$

By Lemma 7.4, there exist integers $M_1 > 0, \dots, M_p > 0$ such that $(M_j \lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for each $j = 1, \dots, p$. Then, for $M = \prod_{j=1}^{p} M_j$, we have $(M\lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for $j = 1, \dots, p$. Thus,

$$e = (0, \mathrm{Id}) = (M\bar{v}, \mathrm{Id}) = \left(\sum_{j=1}^{p} M\lambda_j g_j v, \mathrm{Id}\right) = \prod_{j=1}^{p} (M\lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}.$$

By Theorem 2.7, the system Γ is controllable on G.

7.1.3. The rank condition and irreducible actions. A particular case covered by Theorem 7.1 is the case where K acts irreducibly on V. The following theorem deals with this case and gives a criterion that ensures that $\mathrm{Lie}(\Gamma) = L$ for a given subset Γ of L. To this end, we consider the following construction.

Since the Lie group K acts linearly on the vector space V, the group $G = V \ltimes K$ acts affinely on V:

$$(v, k)x = kx + v, \qquad (v, k) \in G,\ x \in V.$$

For each $M = (v, A) \in L$ and for each $x \in V$, $\{(\exp tM)x \mid t \in \mathbb{R}\}$ is a one-parameter group on V whose infinitesimal generator in the affine vector field $x \mapsto Ax + v$.

Definition 7.2. Given a subset $\Gamma \subset L$, then $\mathcal{F}(\Gamma)$ is the set of affine vector fields on V induced by Γ, i.e., $X \in \mathcal{F}(\Gamma)$ if and only if $X(x) = Ax + v$ for some $(v, A) \in \Gamma$.

We denote by $\mathcal{F}_x(\Gamma)$ the set $\{X(x) \mid X \in \mathcal{F}(\Gamma)\}$. Then we have the following assertion.

Theorem 7.2. *Assume that K is a connected, compact, semisimple real Lie group that acts linearly and irreducibly on a vector space V. Let $G = V \ltimes K$, and let $\Gamma \subset L$. Then a necessary and sufficient condition for* $\mathrm{Lie}(\Gamma) = L$ *is that*

(i) $\mathrm{Lie}(\Gamma_K) = \mathrm{Lie}(\pi_*(\Gamma)) = L(K)$;

(ii) $\mathcal{F}_x(\Gamma) \neq \{0\}$ *for all* $x \in V$.

Proof. Denote by $\mathcal{O}(\mathcal{F})(x)$ the orbit of $\mathcal{F}(\Gamma)$ passing through $x \in V$, i.e., the action of the group generated by $\{\exp tX \mid t \in \mathbb{R},\ X \in \mathcal{F}(\Gamma)\}$. Let H denote the orbit $\mathcal{O}_\Gamma$, i.e., the subgroup of G generated by $\{\exp tA \mid t \in \mathbb{R},\ A \in \Gamma\}$. Then $\mathcal{O}(\mathcal{F})(x) = Hx$.

If $\mathrm{Lie}(\Gamma) = L$, then $H = G$, since G is connected. Thus, the orbits of $\mathcal{F}(\Gamma)$ passing through each point $x \in V$ are given by Gx. But $Gx \neq x$ for any $x \in V$; therefore, for each $x \in V$, there exists $X \in \mathcal{F}(\Gamma)$ such that $X(x) \neq 0$. That is, condition (ii) holds. Since condition (i) is obviously satisfied, the necessity follows.

To prove the sufficiency, assume that (i) and (ii) hold. Let π_Γ be the restriction of the projection π to $\mathrm{Lie}(\Gamma)$. Thus, $\pi_\Gamma : \mathrm{Lie}(\Gamma) \to L(K)$ is a Lie algebra homomorphism. By condition (i), π_Γ is onto. Since $\ker \pi_\Gamma$ is an ideal of $\mathrm{Lie}(\Gamma)$ and since π_Γ is onto, it follows that $\ker \pi_\Gamma$ is a linear subspace of V that is invariant under K. By the irreducibility assumption, either $\ker \pi_\Gamma = V$ or $\ker \pi_\Gamma = \{0\}$.

If $\ker \pi_\Gamma = V$, then, obviously, $\mathrm{Lie}(\Gamma) = L$. To complete the proof, we show that the case $\ker \pi_\Gamma = \{0\}$ is impossible. If $\ker \pi_\Gamma = \{0\}$, then $\mathrm{Lie}(\Gamma)$ is isomorphic to $L(K)$. Since K is semisimple and compact, it follows that the integral group H of $\mathrm{Lie}(\Gamma)$ is compact. For any $x \in V$, the mean $\bar{x} = \int_H hx\, d\mu$

is a fixed point of H (μ is a normalized Haar measure on H). Then $\mathcal{F}_{\bar{x}} = 0$; this contradicts assumption (ii). Thus, $\ker \pi_\Gamma \neq \{0\}$, and the proof is complete.

7.2. K is compact and has nonzero fixed points in V. If the linear action of a compact Lie group K has nonzero fixed points in V, then the rank condition is no longer sufficient for controllability.

Example 7.1. Let $K = \mathrm{SO}(1;\mathbb{R}) \times \mathrm{SO}(n;\mathbb{R})$, and let $V = \mathbb{R} \times \mathbb{R}^n$. The compact connected Lie group K acts naturally on the vector space V:

$$(1, g)(x, y) = (x, gy), \quad (1, g) \in K, \ (x, y) \in V.$$

For this action, $Kv = v$ if and only if $v = (x, 0)$.

We take the Lie group $G = V \ltimes K$ and the right-invariant system on it:

$$\Gamma = \{(v, A) \mid v = (x, y), \ x > 0, \ A \in L(K)\}.$$

Then,

(i) $\mathrm{Lie}(\Gamma) = L$ and

(ii) $\mathbb{A} = \{(v, g) \mid v = (x, y), \ x > 0, \ g \in K\}$.

Hence, Γ is not controllable even though it has a full rank.

Now we obtain controllability conditions for the case where the action of a compact connected group K has nonzero fixed points in a vector space V. Denote by $\langle \cdot, \cdot \rangle$ the inner product on V that is invariant under K, and let d_V be the corresponding metric on V. If d_K denotes the left- and right-invariant metric on K, we let d_G to be the corresponding direct product metric on $G = V \ltimes K$:

$$d_G((v_1, g_1), (v_2, g_2)) = d_K(g_1, g_2) + d_V(v_1, v_2), \qquad (v_1, g_1), (v_2, g_2) \in G.$$

If $(w, h) \in G$, then

$$\begin{aligned} d_G((w, h)(v_1, g_1), (w, h)(v_2, g_2)) &= d_G((w + hv_1, hg_1), (w + hv_2, hg_2)) \\ &= d_K(hg_1, hg_2) + d_V(hv_1, hv_2) = d_K(g_1, g_2) + d_V(v_1, v_2). \end{aligned}$$

Thus, d_G is left-invariant.

We denote

$$\begin{aligned} V_1 &= \{v \in V \mid Kv = v\}, \\ V_2 &= V_1^{\perp}. \end{aligned}$$

It follows from the definitions of the subspace V_1 that for any $X \in L(K)$ and for any $v \in V_1$, we have $Xv = 0$. Moreover, if $X \in L(K)$ and $w \in V_2$, then

$$\langle v, Xw \rangle = -\langle Xv, w \rangle = 0 \qquad \text{for all } v \in V_1.$$

Thus, both V_1 and V_2 are invariant under elements of $L(K)$. Let P denote the orthogonal projection of V onto V_1. Recall that τ is the canonical projection of

G onto V; thus, τ_* is the projection of L onto V. Denote by Γ_V the projection $\tau_*(\Gamma)$ of a right-invariant system $\Gamma \subset L$. We now have the following

Theorem 7.3. *Let a compact connected Lie group K act linearly on a vector space V. Then a right-invariant system $\Gamma \subset L$ is controllable on the Lie group $G = V \ltimes K$ if and only if*

(i) $\mathrm{Lie}(\Gamma) = L$ *and*

(ii) *the convex cone spanned by $P(\Gamma_V)$ is equal to V_1.*

Proof. We first prove the necessity. If $(a, A) \in \Gamma$, then $(a, A) = (a_1, 0) \oplus (a_2, A)$, where $a_1 = Pa$ and $a_2 = a - a_1$; the sign $\oplus$ means that the elements $(a_1, 0)$ and (a_2, A) commute. Hence

$$\exp t(a, A) = (a_1 t, \mathrm{Id})(a_2(t), \exp tA), \quad \text{where} \quad a_2(t) \in V_2 \text{ for all } t$$

since $AV_2 \subset V_2$.

It is now clear that if $Y = (b, B)$ is one more element of Γ, then

$$\begin{aligned}&\exp t_2(b, B) \cdot \exp t_1(a_1, A)\\ &\qquad = (a_1 t_1 + b_1 t_2, \mathrm{Id})(b_2(t_2) + (\exp t_2 B) a_2(t), \exp t_2 B \cdot \exp t_1 A).\end{aligned}$$

Thus, the projection of $\mathbb{A}$ onto V_1 is equal to the convex cone spanned by $P(\Gamma_V)$. If Γ is controllable, then such a cone should be equal to V_1.

To prove the sufficiency, assume that $\mathrm{Lie}(\Gamma) = L$ and $\mathrm{co}(P(\Gamma_V)) = V_1$.

Let

$$T_\Gamma = \{(v, \mathrm{Id}) \mid (v, \mathrm{Id}) \in \mathrm{int}\, \mathbb{A}\}$$

as above. By Lemma 7.2, the set T_Γ is nonempty. If $(z, \mathrm{Id}) \in T_\Gamma$, then let $w = \int_K Kz\, d\mu$, where μ is a normalized Haar measure on K. We have $Kw = w$; hence, $w \in V_1$. If $w = 0$, then, as in the proof of Theorem 7.1, it follows that $(0, \mathrm{Id}) \in T_\Gamma$ and $\mathbb{A} = G$.

If $w \neq 0$, then there exists a positive integer N such that $(v, \mathrm{Id}) \in T_\Gamma$ for $v = Nw$. Indeed, w belongs to the convex cone spanned by the orbit Kv. Thus, $w = \sum_{j=1}^{p} \lambda_j g_j v$ for some elements $g_1 \ldots, g_p$ in K and positive numbers $\lambda_1, \ldots, \lambda_p$. By Lemma 7.4, there exist integers M_j such that $(M_j \lambda_j g_j v, \mathrm{Id}) \in T_\Gamma$. The required integer N can then be taken to be equal to $\prod_{j=1}^{p} M_j$.

Now we show that there exists $\lambda > 0$ such that both λv and $-\lambda v$ belong to $\mathrm{int}\, \mathbb{A}_{\mathrm{co}(\Gamma)}$. Since $\mathrm{co}(P(\Gamma_V)) = V_1$, there exists an element of $\mathrm{co}(\Gamma)$ of the form $X = (-v + u, A)$, where $u \in V_2$ and $A \in L(K)$, and

$$\exp tX = \exp t(-v + u, A) = (-vt + u(t), \exp tA), \qquad \text{where } u(t) \in V_2 \text{ for all } t.$$

Since $(v, \mathrm{Id}) \in T_\Gamma \subset \mathrm{int}\, \mathbb{A}$, it follows that some ball $B((v, \mathrm{Id}), \varepsilon)$ of radius ε centered at (v, Id) is contained in $\mathrm{int}\, \mathbb{A}$. From the left-invariance of the metric d_G, it follows that $B((\exp tX)(v, \mathrm{Id}), \varepsilon)$ is contained in $\mathrm{int}\, \mathbb{A}_{\mathrm{co}(\Gamma)}$. Now K is compact; hence, there exists time $t > 1$ such that $d_K(\exp tA, \mathrm{Id}) < \varepsilon$. Therefore,

$$d_G(((1-t)v + u(t), \mathrm{Id}), ((1-t)v + u(t), \exp tA)) < \varepsilon.$$

Thus,

$$(\exp tX)(v, \mathrm{Id}) = ((1-t)v + u(t), \mathrm{Id}) \in B((\exp tX)(v, \mathrm{Id}), \varepsilon),$$

and hence,

$$((1-t)v + u(t), \mathrm{Id}) \in T_{\mathrm{co}(\Gamma)}.$$

Now, $\int_K K((1-t)v + u(t))\, d\mu = (1-t)v$, and by a preceding argument, it follows that $(M(1-t)v, \mathrm{Id}) \in T_{\mathrm{co}(\Gamma)}$ for some positive integer M. Since $M(1-t) < 0$, it follows from Lemma 7.3, that there exists a sufficiently large $\lambda > 0$ such that both λv and $-\lambda v$ are in $T_{\mathrm{co}(\Gamma)}$. Since $T_{\mathrm{co}(\Gamma)}$ is a semigroup, it follows that $(0, \mathrm{Id}) = (\lambda v, \mathrm{Id}) \cdot (-\lambda v, \mathrm{Id})$ is in $T_{\mathrm{co}(\Gamma)}$. Thus, $T_{\mathrm{co}(\Gamma)}$ contains the identity of G. This shows us that $\mathbb{A}_{\mathrm{co}(\Gamma)} = \Gamma$. But $\mathbb{A}_{\mathrm{co}(\Gamma)} \subset \mathrm{cl}\, \mathbb{A}$; consequently, $\mathrm{cl}\, \mathbb{A} = G$. Together with the assumption $\mathrm{Lie}(\Gamma) = L$, this implies that the system Γ is controllable; see Theorem 2.8.

7.3. Semidirect product of a vector space with an arbitrary Lie group.

Theorem 7.4. *Let H be a connected Lie group that acts linearly on a finite-dimensional real vector space V, and let $G = V \ltimes H$. Assume that H contains a compact group K that has no nonzero fixed points in V. Then a necessary and sufficient condition for a right-invariant system $\Gamma \subset L$ to be controllable on G is that*

(i) $\mathrm{Lie}(\Gamma) = L$ *and*

(ii) $\Gamma_H = \pi_*(\Gamma)$ *is controllable on H.*

Proof. The conditions of the theorem are obviously necessary. To prove the sufficiency, assume that conditions (i) and (ii) hold. By Theorem 2.4, the full-rank system Γ is accessible, i.e., $\mathrm{int}\, \mathbb{A}$ is nonempty. If $(v, g) \in \mathrm{int}\, \mathbb{A}$, then, by condition (ii), there exists $w \in V$ such that $(w, g^{-1}) \in \mathbb{A}$. Thus, $(v, g) \cdot (w, g^{-1}) = (v + gw, \mathrm{Id}) \in \mathrm{int}\, \mathbb{A}$. Hence the set T defined by (7.2) is nonempty. If $(v, \mathrm{Id}) \in T$, then the element $w = \int_K Kv\, d\mu$ is invariant under K, and hence, $w = 0$. The rest of the proof is the same as in the proof of Theorem 7.1. Hence, $e = (0, \mathrm{Id}) \in T \subset \mathrm{int}\, \mathbb{A}$, and thus, $\mathbb{A} = G$ by Theorem 2.7.

The following example shows us that without any assumption on the compact subgroup K, conditions (i) and (ii) do not in general guarantee the controllability.

Example 7.2. Let $H = \mathrm{SO}_0(n, 1)$ be the connected component passing through the identity of the Lorentz group in $\mathbb{R}^n$. This group, as a subgroup of $\mathrm{GL}(n+1; \mathbb{R})$, acts linearly on $V = \mathbb{R}^{n+1}$. Consider the Lie group $G = V \ltimes H$. Let C be the light cone of H in V, and let $\Gamma = C \rtimes L(H)$. Then conditions (i) and (ii) are satisfied, but the attainable set is $\mathbb{A} = C \ltimes H \neq G$. In this case, the maximal compact subgroup K of H is equal to $\mathrm{SO}(n;\mathbb{R}) \times \mathrm{SO}(1;\mathbb{R})$, which has many fixed points in V.

7.4. Homogeneous spaces.

7.4.1. Serret–Frenet frames in $\mathbb{R}^3$. The Serret–Frenet system associated with a curve $x(t)$ in $\mathbb{R}^3$ (see Sec. 6.2.3) is given by

$$\frac{dx}{dt} = R(t)e_1, \qquad \frac{dR}{dt} = R(t)\begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

If both the curvature k and the torsion τ are constant, then

$$\omega = \begin{pmatrix} \tau \\ 0 \\ k \end{pmatrix}$$

is the axis of rotation for

$$A = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Then $\exp tA$ is the rotation about ω by the angle $t\sqrt{\tau^2 + k^2}$, and $x(t)$ is a helix along ω.

Assume now that we consider curves whose curvature $k = \text{const} \neq 0$ and whose torsion can take two distinct values, τ_1 and τ_2. Such curves are concatenations of helices along

$$\omega_1 = \begin{pmatrix} \tau_1 \\ 0 \\ k \end{pmatrix} \quad \text{and} \quad \omega_2 = \begin{pmatrix} \tau_2 \\ 0 \\ k \end{pmatrix}.$$

The corresponding family of left-invariant vector fields on the Euclidean group $G = \mathrm{E}(3;\mathbb{R}) = \mathbb{R}^3 \ltimes \mathrm{SO}(3;\mathbb{R})$ is $\Gamma = \{(e_1, A), (e_1, B)\} \subset \mathfrak{e}(3;\mathbb{R}) = \mathbb{R}^3 \rtimes \mathfrak{so}(3;\mathbb{R})$ with

$$A = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau_1 \\ 0 & \tau_1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau_2 \\ 0 & \tau_2 & 0 \end{pmatrix}.$$

It follows that $\mathrm{Lie}(\Gamma) = \mathbb{R}^3 \rtimes \mathfrak{so}(3;\mathbb{R})$ because of the following calculations:

$$(e_1, A) - (e_1, B) = (\tau_1 - \tau_2)(0, A_1)$$

and

$$[(e_1, A), (e_1, B)] = (\tau_1 - \tau_2)(0, A_2),$$

where we denote

$$A_1 = E_{32} - E_{23}, \quad A_2 = E_{13} - E_{31}, \quad A_3 = E_{21} - E_{12}.$$

Then $[(0, A_1), (0, A_2)] = (0, A_3)$, and therefore, $(0, \mathfrak{so}(3;\mathbb{R})) \subset \mathrm{Lie}(\Gamma)$. Hence we obtain $(e_1, 0) \in \mathrm{Lie}(\Gamma)$, and then, $[(e_1, 0), (0, \mathfrak{so}(3;\mathbb{R}))] = (\mathbb{R}^3, 0) \subset \mathrm{Lie}(\Gamma)$. Thus, $\mathrm{Lie}(\Gamma) = \mathbb{R}^3 \rightthreetimes \mathfrak{so}(3;\mathbb{R}) = \mathfrak{e}(3;\mathbb{R})$.

According to Theorem 7.1, any initial point $x_0 \in \mathbb{R}^3$ and any initial frame at x_0 can be connected with any terminal point $x_1 \in \mathbb{R}^3$ and any terminal frame at x_1 along the integral curves of the left-invariant family Γ in $G = \mathrm{E}(3;\mathbb{R})$.

7.4.2. Serret–Frenet frames in $\mathbb{R}^n$. Results of the previous subsubsection are generalized to curves in $\mathbb{R}^n$ that have all curvatures, except for one, to be fixed, while the remaining free curvature can take any positive value. Indeed, according to Theorem 6.3, the matrices A and B that correspond to such a case generate either $\mathfrak{so}(n;\mathbb{R})$ or $\mathfrak{u}(2m;\mathbb{R})$. The corresponding control system in G is given by

$$\Gamma = \{(e_1, A + uB) \mid u > 0\}.$$

We will show now that $\mathrm{Lie}(\Gamma) = \mathbb{R}^n \rightthreetimes L(K)$ with $L(K)$ equal to the Lie algebra of either $K = \mathrm{SO}(n;\mathbb{R})$ or $K = \mathrm{U}(2m;\mathbb{R})$. By Theorem 6.3, the projection

$$\pi_\Gamma : \mathrm{Lie}(\Gamma) \to L(K), \qquad (a, A) \mapsto A,$$

is onto. On the other hand, $\ker \pi_\Gamma$ cannot be equal to zero, since otherwise $\mathrm{Lie}(\Gamma)$ would be isomorphic to $L(K)$; this is impossible, since, for the system Γ, $\mathbb{A}$ is not contained in any compact subgroup of $G = \mathbb{R}^n \ltimes K$. Thus, $(a, 0) \in \mathrm{Lie}(\Gamma)$ for some $a \in \mathbb{R}^n$, $a \neq 0$. The multiplication rule in $\mathbb{R}^n \rightthreetimes L(K)$,

$$[(v, 0), (b, B)] = (-Bv, 0),$$

implies that $\ker \pi_\Gamma$ is an ideal in L. Taking into account that $\mathrm{Im}\, \pi_\Gamma = L(K)$ and $L(K)a = \mathbb{R}^n$, we obtain $\mathrm{Lie}(\Gamma) = L$.

Therefore, Theorem 7.1 is applicable, and the corresponding controllability conclusions for curves in $\mathbb{R}^n$ follow as in the previous subsubsection.

7.4.3. Affine systems on $\mathbb{R}^n$. Consider the single-input affine system

$$\dot{x} = Ax + a + u(Bx + b), \qquad x \in \mathbb{R}^n, \ u \in U \subset \mathbb{R}, \tag{7.3}$$

where A and B are real $n \times n$ matrices; a and b are vectors in $\mathbb{R}^n$.

Equation (7.3) can be regarded as a part of a larger system defined as follows. Denote by H the orbit of the right-invariant system

$$\{A + uB \mid u \in U\} \subset \mathfrak{gl}(n;\mathbb{R}) \tag{7.4}$$

in $\mathrm{GL}(n;\mathbb{R})$. The elements $X = (a, A)$ and $Y = (b, B)$ belong to the Lie algebra $L = \mathbb{R}^n \rightthreetimes L(H) \subset \mathfrak{aff}(n;\mathbb{R})$ of the Lie group $G = \mathbb{R}^n \ltimes H \subset \mathrm{Aff}(n;\mathbb{R})$; we denote by

$L(H)$ the Lie algebra of H; it is the subalgebra of $\mathfrak{gl}(n;\mathbb{R})$ generated by set (7.4). Thus,

$$\Gamma = \{X + uY \mid u \in U\} \subset L$$

can be considered as a right-invariant control system on G. The affine system (7.3) is induced by the system Γ. Moreover, the affine action of the Lie group G is transitive on $\mathbb{R}^n$, since G contains all translations. By Corollary 3.3, if the right-invariant system Γ is controllable on the Lie group G, then the affine system (7.3) is controllable on $\mathbb{R}^n$.

By construction, for the system $\Gamma_H = \pi_*(\Gamma)$ projected onto H, we have $\operatorname{Lie}(\Gamma_H) = L(H)$. That is why, by Theorem 7.4, the right-invariant system Γ (and consequently, the affine system (7.3)) is controllable if

(i) either H is compact, or Γ_H is controllable on H, and

(ii) $\operatorname{Lie}(\Gamma) = L$.

7.5. Remarks. The results of Secs. 7.1–7.3 are due to Bonnard, Jurdjevic, Kupka, and Sallet [28].

The proof of Theorem 7.1 for the particular cases $G = \mathbb{R}^n \ltimes \mathrm{SO}(n;\mathbb{R})$ and $\mathbb{R}^{2m} \ltimes \mathrm{U}(2m;\mathbb{R})$ in Sec. 7.1.1 and the applications in Sec. 7.4 were developed by Jurdjevic [79].

One of the early results on controllability of right-invariant systems on the Euclidean group was obtained by Sallet [124]. This proposition obviously follows from Theorem 7.1:

Theorem 7.5. *Let* $X_1 = (a, A)$, $X_2 = (b, B) \in \mathbb{R}^n \rtimes \mathfrak{so}(n;\mathbb{R})$ *be right-invariant vector fields on the Lie group* $G = \mathrm{E}(n;\mathbb{R})$. *Then a sufficient condition for the system* $\Gamma = \{X_1, X_2\}$ *to be controllable on* G *is*

(i) $\operatorname{Lie}(X_1, X_2) = L$ *and*

(ii) $a \in \operatorname{Im} A$ *and* $b \in \operatorname{Im} B$.

8. Semisimple Lie Groups

A Lie algebra L is called *semisimple* if it contains no nonzero solvable ideals. A Lie group G is called *semisimple* if its Lie algebra L is semisimple. A Lie algebra L is called *simple* if it contains no nontrivial (i.e., distinct from $\{0\}$ and L) ideals. A semisimple Lie algebra is a direct sum of its simple non-abelian ideals.

In this section, we assume that L is a real finite-dimensional semisimple Lie algebra.

8.1. Preliminaries.

8.1.1. Regular elements. For any element $B \in L$, the adjoint operator

$$\operatorname{ad} B : L \to L, \quad \operatorname{ad} B(C) = [B, C], \ C \in L,$$

is defined. A Lie algebra L is semisimple if and only if the *Killing form*

$$\operatorname{Kil} : L \times L \to \mathbb{R}, \quad \operatorname{Kil}(A, B) = \operatorname{tr}(\operatorname{ad} A \circ \operatorname{ad} B)$$

is nondegenerate.

The roots of the characteristic polynomial

$$P_B(t) = \det(\operatorname{ad} B - t \operatorname{Id}) = (-1)^n t^n + a_1(B) t^{n-1} + a_2(B) t^{n-2} + \cdots + a_n(B),$$
$$n = \dim L,$$

are eigenvalues of the operator $\operatorname{ad} B$, $B \in L$, and $a_1(B), \ldots, a_n(B)$ are forms on L. Since $\operatorname{ad} B(B) = 0$, we have $a_n(B) \equiv 0$. The smallest number r such that

$$a_{n-r+1} \equiv 0, \ a_{n-r+2} \equiv 0, \ \ldots, \ a_n \equiv 0, \quad \text{but} \quad a_{n-r} \not\equiv 0,$$

is called the *rank* of the Lie algebra L and is denoted by $\operatorname{rk} L$. An element $B \in L$ is called *regular* if

$$a_{n-r}(B) \neq 0, \quad r = \operatorname{rk} L.$$

For a regular element B, zero $0 \in \mathbb{C}$ is an eigenvalue of the adjoint operator $\operatorname{ad} B$ with the multiplicity r; thus,

$$\dim(\ker \operatorname{ad} B) = \operatorname{rk} L.$$

The set of regular elements is open and dense in L.

8.1.2. Weyl basis and normal real form. Let $\mathcal{L}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Let L_0 be the *Cartan subalgebra* of $\mathcal{L}$, i.e., a nilpotent subalgebra that is its own normalizer in $\mathcal{L}$. Denote by R the set of nonzero roots of $\mathcal{L}$ with respect to L_0. Then there is a decomposition of $\mathcal{L}$ into the direct sum

$$\mathcal{L} = L_0 \oplus \sum\nolimits^{\oplus} \{L_\alpha \mid \alpha \in R\},$$

where L_α, $\alpha \in R$, are root spaces, which are one-dimensional.

For any $\alpha \in R$, there exists a unique element $H_\alpha \in L_0$ such that

$$\operatorname{Kil}(H, H_\alpha) = \alpha(H) \text{ for all } H \in L_0.$$

Define the following subspace of L_0:

$$L(0) = \sum\nolimits^{\oplus} \{\mathbb{R} H_\alpha \mid \alpha \in R\}.$$

We have

$$L_0 = L(0) \oplus i L(0).$$

One can identify R with the dual space $L(0)^*$ of $L(0)$ and then introduce an ordering of roots in R induced by some vector space ordering of $L(0)^*$. A positive root is called fundamental if it cannot be written as a sum of two positive roots. Denote by Δ^+ the set of fundamental roots.

For any root $\alpha \in R$, there exists an element $E_\alpha \in L_\alpha$ such that $\mathrm{Kil}(E_\alpha, E_{-\alpha}) = 1$, and for all $\alpha, \beta \in R$,

$$
\begin{aligned}
&[E_\alpha, E_{-\alpha}] = H_\alpha, \\
&[H, E_\alpha] = \alpha(H) E_\alpha \quad \text{for all } H \in L_0, \\
&[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin R, \\ N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in R, \end{cases}
\end{aligned}
$$

where $N_{\alpha\beta}$ are some real constants. The system

$$H_\alpha,\ \alpha \in \Delta^+, \qquad E_\alpha,\ \alpha \in R$$

is called a *Weyl basis* of $\mathcal{L}$ with respect to L_0.

The subspace

$$L = L(0) \oplus \sum\nolimits^{\oplus} \{\mathbb{R} E_\alpha \mid \alpha \in R\} \tag{8.1}$$

is a *normal real form* of the complex Lie algebra $\mathcal{L}$; it is unique up to an isomorphism.

8.1.3. Strongly regular elements. Any element $A \in L$ admits a unique decomposition

$$A = A(0) + \sum \{k_\alpha E_\alpha \mid \alpha \in R\}, \tag{8.2}$$

where

$$A(0) \in L(0), \quad k_\alpha \in \mathbb{R}.$$

Remark. An element $B \in L(0)$ is regular if and only if the elements $\alpha(B)$ are nonzero for all $\alpha \in R$.

It turns out that the following two variations of the regularity property are relevant in controllability questions.

Definition 8.1. An element $B \in L$ is called *strongly regular* if

(i) B is regular and

(ii) every nonzero eigenvalue of $\operatorname{ad} B$ is simple.

Definition 8.2. Given $A \in L$ with $A \notin L(0)$ and $B \in L(0)$, the element B is called A-*strongly regular* if the elements $\alpha(B)$ are nonzero and distinct for all $\alpha \in R$ such that $k_\alpha \neq 0$ in decomposition (8.2).

Remark. To compare strong regularity and A-strong regularity, we notice that an element $B \in L(0)$ is strongly regular if and only if the elements $\alpha(B)$ are nonzero and distinct for *all* roots $\alpha \in R$.

8.1.4. Root decompositions along eigenspaces of a strongly regular element. Choose and fix a strongly regular element $B \in L$.

The complexification $L_c = L \otimes_{\mathbb{R}} \mathbb{C}$ is a complex semisimple Lie algebra. The adjoint operator in L_c is defined by

$$\operatorname{ad}_c B \,:\, L_c \to L_c, \quad \operatorname{ad}_c B(C) = [B, C],\; C \in L_c.$$

By the strongly regular property of B, the space

$$L_0 = \ker \operatorname{ad}_c B$$

is the Cartan subalgebra of L_c. Denote by $\operatorname{Sp}(B)$ the subset of $\mathbb{C}$ of all *nonzero* eigenvalues of $\operatorname{ad} B$. Notice that there is an isomorphism

$$R \to \operatorname{Sp}(B), \qquad \alpha \mapsto \alpha(B) \tag{8.3}$$

between $\operatorname{Sp}(B)$ and R, the set of roots of L_c with respect to the Cartan subalgebra L_0. That is why we can denote the root spaces

$$L_\alpha, \quad \alpha \in R,$$

by

$$L_a, \quad a = \alpha(B) \in \operatorname{Sp}(B).$$

Notice that $L_{\overline{a}} = \sigma(L_a)$, $a \in \operatorname{Sp}(B)$, where $\overline{a}$ is the complex conjugate to an eigenvalue a and σ is the conjugation in L_c with respect to L.

For $a \in \operatorname{Sp}(B)$, consider the real space

$$L(a) = (L_a + L_{\overline{a}}) \cap L.$$

Notice that

$$\dim L(a) = 1 \text{ if } a \in \mathbb{R};$$

in this case, $L(a)$ is the eigenspace of $\operatorname{ad} B$ corresponding to the eigenvalue a, and

$$\dim L(a) = 2 \text{ if } a \notin \mathbb{R};$$

then $L(a)$ is an invariant subspace of $\operatorname{ad} B$. Thus, we obtain the following decompositions into direct sums of eigenspaces and invariant spaces:

$$\begin{aligned} L_c &= \ker \operatorname{ad}_c B \oplus {\textstyle\sum^{\oplus}} \{L_a \mid a \in \operatorname{Sp}(B)\}, \\ L &= \ker \operatorname{ad} B \oplus {\textstyle\sum^{\oplus}} \{L(a) \mid a \in \operatorname{Sp}(B),\ \operatorname{Im} a \ge 0\}. \end{aligned} \tag{8.4}$$

Then any element $A \in L_c$ has a complex decomposition

$$A = A_0 + \sum \{A_a \mid a \in \operatorname{Sp}(B)\}, \quad A_0 \in \ker \operatorname{ad}_c B,\ A_a \in L_a, \tag{8.5}$$

and any $A \in L$ has a real decomposition

$$\begin{aligned} &A = A(0) + \sum \{A(a) \mid a \in \operatorname{Sp}(B),\ \operatorname{Im} a \ge 0\}, \\ &A(0) \in \ker \operatorname{ad} B,\ A(a) \in L(a). \end{aligned} \tag{8.6}$$

8.2. Homogeneous systems. Now we turn to controllability conditions for right-invariant systems on semisimple Lie groups.

First of all, in semisimple Lie algebras, the rank condition is generically satisfied.

Theorem 8.1. *If L is semisimple, then the set S of all pairs (A, B) in $L \times L$ for which the Lie algebra generated by A, B is equal to L is an open and dense subset of $L \times L$.*

Proof. If $\mathrm{Lie}(A, B) = L$, then the homogeneous system $\{\pm A, \pm B\}$ is controllable on G. But controllable right-invariant systems remain controllable under small perturbations (see Theorem 2.10); thus S is open.

To show that S is dense, take any strongly regular element $B \in L$ and any element A in L for which $A(a) \neq 0$ for $a \in \mathrm{Sp}(B)$ in decomposition (8.6). Such pairs form a dense subset of $L \times L$, and each pair (A, B) belongs to S.

In the semisimple case, homogeneous systems are naturally treated more easily, as well as in the general case (see Sec. 5).

Theorem 8.2. *Let G be a semisimple connected Lie group. Then, for a generic pair of elements A and B in L, the system $\Gamma = \{\pm A, \pm B\}$ is controllable on G.*

Proof. For a generic pair of elements A and B in the Lie algebra of G, the elements A and B generate this Lie algebra; see Theorem 8.1. By Theorem 5.2, the homogeneous system Γ is controllable.

8.3. Multiple-input nonhomogeneous systems. In the case of unbounded control range, the result for multiple-input nonhomogeneous systems is an easy consequence of the proposition from the previous subsection on homogeneous systems.

Theorem 8.3. *Let G be a semisimple connected Lie group. Then, for generic elements A, B_1, …, $B_m \in L$, the system $\Gamma = \{A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R}\}$ is controllable on G.*

Proof. The vector space $\mathrm{span}(B_1, \ldots, B_m)$ is contained in the Lie saturate $\mathrm{LS}(\Gamma)$. By Theorem 8.1, the set of all m-tuples of right-invariant vector fields $(B_1, \ldots, B_m)$ that generate L is open and dense. Each system Γ with such $B_1, \ldots, B_m$ is controllable independently of the drift vector field A.

8.4. Single-input nonhomogeneous systems. Now we consider a much more complicated case of systems of the form $\Gamma = A + \mathbb{R}B$.

8.4.1. Statement of theorems. We endow the complex plane $\mathbb{C}$ with the lexicographic ordering: $a > b$ if $\mathrm{Re}\, a > \mathrm{Re}\, b$ or $\mathrm{Re}\, a = \mathrm{Re}\, b$ and $\mathrm{Im}\, a > \mathrm{Im}\, b$.

Definition 8.3. An eigenvalue $a \in \mathrm{Sp}(B)$ is called *maximum* (resp. *minimum*) if, for any $b \in \mathrm{Sp}(B)$, $b > 0$ (resp. $b < 0$), we have $[L_a, L_b] = \{0\}$.

($[L_a, L_b]$ is the vector space generated by the brackets $[X, Y]$, $X \in L_a$, $Y \in L_b$.)

Theorem 8.4. *Let G be a semisimple connected Lie group with a finite center and Lie algebra L. Then, for $A, B \in L$, the system $\Gamma = A + \mathbb{R}B$ is controllable on G if the following conditions hold*:

(1) *B is strongly regular*;

(2) $\mathrm{Lie}(A, B) = L$;

(3) *let $A = A_0 + \sum\{A_a \mid a \in \mathrm{Sp}(B)\}$ be the decomposition of A along the eigenspaces of $\mathrm{ad}_c B$; see (8.5). Then $A_s \neq 0$ if s is either maximum or minimum*;

(4) *if $s \in \mathrm{Sp}(B)$ is maximum and the real part $r = \mathrm{Re}\, s$ is a nonzero eigenvalue of $\mathrm{ad}_c B$, then $\mathrm{Kil}(A_r, A_{-r}) < 0$ provided that L_r and L_s belong to the same simple ideal of L_c.*

Remark. All the conditions (1)–(4) define semialgebraic subsets of $L \times L$. Moreover, the subsets defined by (1)–(3) are open and dense in $L \times L$.

To approach the proof of Theorem 8.4, consider systems $\Gamma \subset L$ satisfying the following conditions:

(A) Γ is a wedge, i.e., a closed convex positive cone;

(B) the largest vector subspace $E(\Gamma)$ contained in Γ, which is called the edge of Γ, is a Lie subalgebra of L;

(C) for any $X \in E(\Gamma)$ and for any $t \in \mathbb{R}$, $\exp(t\, \mathrm{ad}\, X)$ maps Γ into itself;

(D) $E(\Gamma)$ contains a strongly regular element B;

(E) if $s \in \mathrm{Sp}(B)$ and s is maximum (resp. minimum), then there exists a vector X_+ (resp. X_-) in Γ such that $X_+(s) \neq 0$ (resp. $X_-(s) \neq 0$);

(F) if $r \in \mathrm{Sp}(B)$ is the real part of a maximum eigenvalue s and if L_r and L_s belong to the same simple ideal of L_c, then there exist $X_+, X_- \in \Gamma$ such that $\mathrm{Kil}(X_+(r), X_-(-r)) < 0$.

Remark. Hypotheses (A), (B), and (C) mean that Γ is a Lie wedge; see Sec. 4.

Theorem 8.5. *If a system $\Gamma \subset L$ satisfies conditions* (A)–(F) *and if* $\mathrm{Lie}(\Gamma) = L$, *then Γ is controllable on G.*

The following more general result holds.

Theorem 8.6. *If a system $\Gamma \subset L$ satisfies conditions* D), E), F), *and if* $\mathrm{Lie}(\Gamma) = L$, *then Γ is controllable on G.*

Proof. The Lie saturate $\mathrm{LS}(\Gamma)$ satisfies the assumptions of Theorem 8.5.

The main controllability result for semisimple Lie groups, Theorem 8.4, easily follows from Theorem 8.5.

Proof. The Lie saturate $\mathrm{LS}(\Gamma)$ satisfies all the hypotheses of Theorem 8.5. For example, to show that $\pm B \in \mathrm{E}(\mathrm{LS}(\Gamma))$, consider the limits $\lim_{u \to \pm\infty}(A + uB)/|u| = \pm B$; by property (A), these limits belong to $\mathrm{LS}(\Gamma)$.

8.4.2. Proof of Theorem 8.5. Let $\mathrm{Sim}(L)$ (resp. $\mathrm{Sim}(L_c)$) denote the set of all simple ideals in L (resp. L_c). Then we have the following direct Lie algebra decompositions:

$$L = \sum\nolimits^{\oplus} \{\Sigma \mid \Sigma \in \mathrm{Sim}(L)\}, \qquad L_c = \sum\nolimits^{\oplus} \{S \mid S \in \mathrm{Sim}(L_c)\}.$$

The conjugation $\sigma : L_c \to L_c$ permutes the elements of $\mathrm{Sim}(L_c)$. The connection between $\mathrm{Sim}(L)$ and $\mathrm{Sim}(L_c)$ is as follows:

(1) $\mathrm{Sim}(L) = \{(S + \sigma S) \cap L \mid S \in \mathrm{Sim}(L_c)\}$;

(2) if $\Sigma \in \mathrm{Sim}(L)$ and Σ_c denotes its complexification, then either $\Sigma_c \in \mathrm{Sim}(L_c)$ (inner case) or $\Sigma_c = S_1 \oplus S_2$, $S_1, S_2 \in \mathrm{Sim}(L_c)$ (outer case). In the outer case, Σ induces a real Lie algebra isomorphism $S_1 \to S_2$ and Σ can be identified with the graph of σ in $S_1 \oplus S_2$, i.e., $\Sigma = \{(x, \sigma(x)) \mid x \in S_1\}$.

Let $B \in L$ be a strongly regular element in L. The Cartan subalgebra $L_0 = \ker \mathrm{ad}_c B$ of L_c splits:

$$L_0 = \sum\nolimits^{\oplus} \{L_0 \cap S \mid S \in \mathrm{Sim}(L_c)\}.$$

Each $L_0 \cap S$ is a Cartan algebra of S. The root system R of L_c is the union of the root systems R_S, $S \in \mathrm{Sim}(L_c)$.

In view of bijection (8.3), we write L_α and $L(\alpha)$ instead of $L_{\alpha(B)}$ and $L(\alpha(B))$. The conjugation σ acts on R as follows: $\sigma(\alpha)(X) = \overline{\alpha(\sigma(X))}$, where $X \in L_c$.

Definition 8.4. We endow $R \cup \{0\}$ with the total order structure pull back of the order structure on $\mathrm{Sp}(B) \cup \{0\}$ by the bijection $\alpha \in R \cup \{0\} \mapsto \alpha(B) \in \mathrm{Sp}(B) \cup \{0\}$.

Definition 8.5. A root $\alpha \in R$ is *maximum* (resp. *minimum*) if $\alpha(B) \in \mathbb{C}$ is maximum (resp. minimum) in the sense of Definition 8.3.

This is equivalent to the classical definition: α is maximum (resp. minimum) if $\alpha + \beta \notin R$ whenever $\beta \in R$ and $\beta > 0$ (resp. $\beta < 0$).

Let $S \in \mathrm{Sim}(L_c)$ be a simple component of L_c and s (resp. $-s$) its maximum (resp. minimum) root.

Definition 8.6. Denote by R'_S the set of roots $\alpha \in R_S$ such that $\alpha + s$ or $\alpha - s$ is a root. R''_S will denote the complement $R_S \setminus (R'_S \cup \{s, -s\})$.

Proposition 8.1. (1) *If $\alpha, \beta \in R'_S$ have the same sign and if $\alpha + \beta \in R_S$, then $\alpha + \beta \in \{s, -s\}$.*

(2) *If $\alpha \in R'_S$, $\beta \in R''_S$ and $\alpha + \beta \in R_S$, then $\alpha + \beta \in R'_S$ and α and $\alpha + \beta$ have the same sign.*

(3) *If $\alpha, \beta \in R''_S$ and $\alpha + \beta \in R_S$, then $\alpha + \beta \in R''_S$.*

Corollary 8.1. (1) *If $\alpha, \beta \in R'_S$ have the same sign, then, for all $\gamma \in R''_S$ such that $\alpha + \beta + \gamma \in R_S$ and either $\alpha + \gamma$ or $\beta + \gamma$ is a root, $\alpha + \beta + \gamma \in \{s, -s\}$.*

(2) *If* $\alpha \in R'_S$, $\beta, \gamma \in R''_S$, $\alpha + \beta + \gamma \in R_S$, *and at least one of three linear forms* $\alpha+\beta$, $\beta+\gamma$, *and* $\alpha+\gamma$ *is a root, then* $\alpha+\beta+\gamma \in R'_S$ *and* α *and* $\alpha+\beta+\gamma$ *have the same sign.*

Now we explain the main idea of the proof of Theorem 8.5 in the case of a simple Lie algebra L.

Denote by L' the Lie algebra generated by $\{L_\alpha \mid \alpha \in R'_S \cup \{\pm s\}\}$. Consider the Lie algebra I generated by elements X of L' such that $\mathbb{R}X \subset \Gamma$.

I is nonempty; if we show that it is an ideal of L, then we obtain $\Gamma = L$.

To prove that I is an ideal, we verify that the generators $X(\alpha)$ of I and $Y(\beta)$ of L are such that $[X(\alpha), Y(\beta)] \in I$. This is mainly obtained on the basis of the properties of the sets of roots R'_S and R''_S given by Proposition 8.1 and Corollary 8.1.

In the case of a semisimple Lie algebra L, the idea of the proof of Theorem 8.5 is analogous.

8.5. The special linear group. The Lie group $G = \mathrm{SL}(n;\mathbb{R})$ is simple and has a trivial center; thus, the results of the previous subsection can be applied.

Take any $A, B \in \mathfrak{sl}(n;\mathbb{R})$ and consider the right-invariant system $\Gamma = A + \mathbb{R}B$ on the group $\mathrm{SL}(n;\mathbb{R})$. The eigenvalues of $\operatorname{ad} B$ are differences of eigenvalues of B. Let $\lambda_1, \ldots, \lambda_n$ denote the (possibly complex) eigenvalues of B. Strongly regular elements in $\mathfrak{sl}(n;\mathbb{R})$ are characterized by the inequalities

$$\lambda_i - \lambda_j \neq \lambda_k - \lambda_l, \quad \{i,j\} \neq \{k,l\}, \ i \neq j.$$

The eigenspace L_a of $\operatorname{ad} B$ corresponding to a real eigenvalue $a = \lambda_i - \lambda_j \in \mathrm{Sp}(B)$ consists of all matrices of the form $b_i \otimes c_j$, where b_i and c_j are respectively eigenvalues of B and its transpose B^{T}, i.e., $Bb_i = \lambda_i b_i$ and $B^{\mathrm{T}} c_j = \lambda_j c_j$. If an eigenvalue $a \in \mathrm{Sp}(B)$ is complex, then $\overline{a}$ is also an eigenvalue and L_a is the two-dimensional vector space spanned by $\mathrm{Re}(b_i \otimes c_j)$ and $\mathrm{Im}(b_i \otimes c_j)$. The eigenspace L_0 that corresponds to a zero eigenvalue consists of all matrices that commute with B.

8.5.1. The special linear group in dimension 2. Consider two examples, which are generic for the group $\mathrm{SL}(2;\mathbb{R})$.

Example 8.1. Let

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of B are $\lambda_1 = i$ and $\lambda_2 = -i$; hence the nonzero eigenvalues of the adjoint operator $\operatorname{ad} B$ are $\mathrm{Sp}(B) = \{\pm i\}$. The unit eigenvectors of B are $b_1 = (e_1 + ie_2)/\sqrt{2}$ and $b_2 = \overline{b}_1 = (e_1 - ie_2)/\sqrt{2}$, where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$. Since $B^{\mathrm{T}} = -B$, it follows that $c_1 = b_2$ and $c_2 = b_1$. Thus,

$$b_1 \otimes c_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \text{ and } b_2 \otimes c_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The linear hull of the matrices

$$\operatorname{Re}(b_1 \otimes c_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \operatorname{Im}(b_1 \otimes c_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the two-dimensional vector space of 2×2 symmetric matrices with zero trace. Hence decomposition (8.4) induced by B is the classic decomposition of matrices into the symmetric part and antisymmetric part.

Now we verify conditions of Theorem 8.4. The matrix B is a strongly regular element of the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$. Condition (3) means that the matrix A has a nonzero symmetric part. Under this assumption, A and B generate $\mathfrak{sl}(2;\mathbb{R})$ as a Lie algebra. Finally, condition (4) is absent. Consequently, by Theorem 8.4, the system $\Gamma = A + \mathbb{R}B$ is controllable on $\mathrm{SL}(2;\mathbb{R})$ if the matrix A is not skew-symmetric. This condition is also necessary for controllability: if $A^{\mathrm{T}} = -A$, then the rank condition for Γ is violated.

Example 8.2. Now consider the case of

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of B are ± 1 and $\operatorname{Sp}(B) = \{\pm 2\}$. The corresponding eigenspaces of $\operatorname{ad} B$ are one-dimensional and are spanned by

$$e_1 \otimes e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } e_2 \otimes e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We use B, $e_1 \otimes e_2$, and $e_2 \otimes e_1$ as the basis for decomposition (8.4) and write any matrix $A = (a_{ij})$ as $A = a_{11}B + a_{12}e_1 \otimes e_2 + a_{21}e_2 \otimes e_1$.

By Theorem 8.4, the system $\Gamma = A + \mathbb{R}B$ is controllable on $\mathrm{SL}(2;\mathbb{R})$ if $a_{12}a_{21} < 0$. On the other hand, if $a_{12}a_{21} \geq 0$, then Γ is not controllable, since, in this case, the bilinear system induced by Γ has invariant quadrants in $\mathbb{R}^2$.

Remark. The preceding two examples are exhaustive for $\mathrm{SL}(2;\mathbb{R})$, since any matrix $B \in \mathfrak{sl}(2;\mathbb{R})$ with a nonzero spectrum is similar to one of the matrices

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \ b \in \mathbb{R} \setminus \{0\}.$$

Using the control scaling $u \mapsto u/b$, any system $\Gamma = A + \mathbb{R}B$ on $\mathrm{SL}(2;\mathbb{R})$ with $\det B \neq 0$ can be reduced to the systems considered in the previous two examples.

Now we return to the general case in $\mathrm{SL}(n;\mathbb{R})$. If a strongly regular element B has a real spectrum, then B can be diagonalized. The eigenspaces of $\operatorname{ad} B$ are then one-dimensional spaces spanned by the matrices $e_i \otimes e_j = E_{ij}$, where $e_1, \ldots,$ e_n is the standard basis in $\mathbb{R}^n$. The maximum eigenvalue of $\operatorname{ad} B$ is the largest difference between the diagonal elements of B, and the minimum eigenvalue is negative. Rearranging the base vectors e_i, we can put the diagonal elements of B in the ascending order $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then condition (4) in Theorem 8.4 takes the form $a_{1n}a_{n1} < 0$, where a_{ij} is the general entry of the drift matrix A.

This argument leads to the following result.

Theorem 8.7. *Let $n \times n$ real matrices $A = (a_{ij})$ and B with zero trace satisfy the conditions*:

(i) $a_{1n}a_{n1} < 0$;

(ii) $B = \mathrm{diag}(b_1, \dots, b_n)$;

(iii) $b_1 < b_2 < \cdots < b_n$;

(iv) $b_i - b_j \neq b_k - b_m$ *for* $(i, j) \neq (k, m)$.

Then the system $\Gamma = A + \mathbb{R}B$ is controllable on the group $\mathrm{SL}(n; \mathbb{R})$ if and only if the matrix A is permutation-irreducible.

Recall that an $n \times n$ matrix A is called *permutation-reducible* if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a $k \times k$ matrix with $0 < k < n$. An $n \times n$ matrix is called *permutation-irreducible* if it is not permutation-reducible. Permutation-irreducible matrices are matrices having no nontrivial invariant coordinate subspaces.

8.5.2. The conjecture of Jurdjevic–Kupka. In the case of strongly regular elements B with real eigenvalues, Theorem 8.4 covers the case of a skew-symmetric drift term A. From this point of view, the case where both A and B are symmetric is at the other end of the spectrum created by Theorem 8.4.

Conjecture 8.1. If matrices $A, B \in \mathfrak{sl}(n; \mathbb{R})$ are symmetric, then the right-invariant system $\Gamma = A + \mathbb{R}B$ is not controllable, neither on $\mathrm{SL}(n; \mathbb{R})$ nor on $\mathbb{R}^n \setminus \{0\}$.

In dimensions $n = 2, 3$, this conjecture is easily proved by constructing invariant quadrants or octants for the induced bilinear system

$$\dot{x} = Ax + uBx, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u \in \mathbb{R}. \tag{8.7}$$

For $n > 3$, the question remains open. A partial confirmation of the preceding conjecture in arbitrary dimensions under some additional assumptions is given by computing all invariant orthants of bilinear systems in $\mathbb{R}^n$.

8.5.3. Invariant orthants of bilinear systems. Let $A, B_1, \dots, B_m$ be arbitrary real $n \times n$ matrices. In this subsubsection, we present a criterion for a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u_i \in \mathbb{R} \tag{8.8}$$

to have invariant orthants. This result implies a partial confirmation of Conjecture 8.1.

First, we give the necessary notation and definitions. The set of indices

$$\Sigma_n = \{ \sigma = (\sigma_1, \ldots, \sigma_n) \mid \sigma_i = \pm 1 \ \forall i = 1, \ldots, n \}$$

will be used for parametrization of *orthants*, i.e., sets of the form

$$\mathbb{R}^n_\sigma = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \sigma_i \geq 0 \ \forall i = 1, \ldots, n \}.$$

A subset of the state space is called *positive* (*negative*) *invariant* for a vector field or a control system if all trajectories of the field or the system starting in this set (resp., its complement) do not leave it (resp., its complement) for all positive instants of time.

Remark. A system is globally controllable iff it has neither positive nor negative invariant sets (except the trivial ones, the whole state space and the empty set). Thus, conditions for existence for nontrivial invariant sets are sufficient conditions for global noncontrollability.

Definition 8.7. An $n \times n$ matrix $A = (a_{ij})$ is called *sign-symmetric* if $a_{ij} a_{ji} \geq 0$ for all $i, j = 1, \ldots, n$.

Construction 8.1. For any sign-symmetric $n \times n$ matrix A, we construct the *graph* $H(A)$ by the following rule. The graph $H(A)$ has n vertices $1, 2, \ldots, n$. Its vertices i, j, $i \neq j$, are connected by the edge (i, j) if and only if at least one of the numbers a_{ij} and a_{ji} is nonzero. We take into account only edges that connect *distinct* vertices of the graph $H(A)$; self-loops are thus explicitly excluded from consideration. Every edge (i, j) is marked by the sign $+$ or $-$: if $a_{ij} \geq 0$ and $a_{ji} \geq 0$, then the sign $+$ is applied, and if $a_{ij} \leq 0$ and $a_{ji} \leq 0$, then we apply $-$ (there can be no other combinations of signs by virtue of sign-symmetry of A). The marked edges are called *positive* or *negative* depending on the sign $+$ or $-$. For the graph $H(A)$, we define the following function $s(i, j)$, $i, j = 1, \ldots, n$, $i \neq j$: $s(i, j) = 0$ if the vertices i, j are not connected by an edge in $H(A)$, $s(i, j) = 1$ for the positive, and $s(i, j) = -1$ for the negative edge (i, j) in the graph $H(A)$. A loop (i.e., a closed path composed of edges) of a graph is called *even* (*odd*) if it contains an even (resp. odd) number of negative edges. A graph *satisfies the even-loop property* if all its loops are even.

Remark. If a graph H satisfies the even-loop property, then there is a subset V of the set of its vertices such that:

(a) any negative edge of the graph H has exactly one vertex in V;

(b) any positive edge of the graph H has either 0 or 2 vertices in V.

In other words, such a graph H is *bichromatic*: its vertices can be colored in two colors so that odd edges connect vertices of distinct colours, and even edges connect vertices of coinciding colours; the first color corresponds to the set V and the second one to its complement.

Construction 8.2. Assume that a graph H satisfies the even-loop property and V is any subset of the set of its vertices that satisfies the previous conditions (a) and (b). Then the *index of the graph H* corresponding to the set V is the set $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_n$ defined as follows: $\sigma_i = +1$ if $i \notin V$ and $\sigma_i = -1$ if $i \in V$.

Theorem 8.8. *Let $A, B_1, \dots, B_m$ be $n \times n$ matrices. The bilinear system* (8.8) *has positive* (*negative*) *invariant orthants if and only if the following conditions hold*:

1. *the matrix A is sign-symmetric*;

2. *the matrices B_1, ..., B_m are diagonal*;

3. *the graph $H(A)$* (*resp. $H(-A)$*) *satisfies the even-loop property.*

Then positive (negative) invariant orthants are $\mathbb{R}^n_\sigma$, where σ is any index of the graph $H(A)$ (resp. $H(-A)$), and their number is equal to 2^c, where c is the number of connected components of the graph $H(A)$.

If system (8.8) has no invariant coordinate subspaces (in particular, if this system satisfies the rank condition everywhere in $\mathbb{R}^n \setminus \{0\}$), then it has either 0 or 2 invariant orthants.

Proof. The outline of the proof of Theorem 8.8 is as follows. System (8.8) has invariant orthants if and only if the matrices B_i, $i = 1, \dots, m$, are diagonal and the linear vector field Ax has invariant orthants. The search for these orthants is based on two ideas. First, it is common knowledge that the positive orthant

$$\mathbb{R}^n_+ = \{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i = 1, \dots, n \,\}$$

is positive invariant for the field Ax if and only if all off-diagonal entries of the matrix A are nonnegative. Second, if the field Ax has an invariant orthant, then successive changes of coordinates $(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, -x_i, \dots, x_n)$ in $\mathbb{R}^n$ map this orthant onto $\mathbb{R}^n_+$. During this process, we can keep track of signs of entries a_{ij} of the matrix A and obtain conditions for existence of invariant orthants in terms of sign combinations of a_{ij}. These conditions are conveniently expressed in terms of the graph $H(A)$ that corresponds to the matrix A as given in Theorem 8.8.

Theorem 8.8 is related to Conjecture 8.1, since there exists an orthogonal transformation of $\mathbb{R}^n$ that diagonalizes a symmetric matrix B; then a symmetric

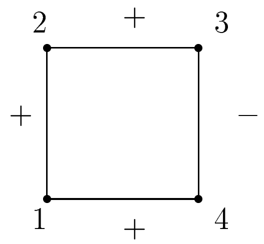

Fig. 1. The graph $H(A)$.

matrix A turns into a symmetric one. That is why we can assume that B is diagonal and A is symmetric.

Theorem 8.8 implies that Conjecture 8.1 holds in dimensions 2 and 3: in fact, for these dimensions, if A is sign-symmetric and B is diagonal, then system (8.7) has a positive or negative invariant orthant. Even for $n = 4$, there are symmetric matrices A for which the field Ax and system (8.7) have no invariant orthants (see the example below). Here the question of global controllability, i.e., of absence of *any* invariant sets, is left open. But for symmetric matrices A with at least one of the graphs $H(A)$, $H(-A)$ satisfying the even-loop property, Conjecture 8.1 is valid. However, in these cases not the symmetry but the sign-symmetry of A is essential.

Example 8.3. Let $A = (a_{ij})$ be any 4×4 matrix of the form

$$\begin{pmatrix} * & + & 0 & + \\ + & * & + & 0 \\ 0 & + & * & - \\ + & 0 & - & * \end{pmatrix},$$

i.e., $a_{12}, a_{21}, a_{14}, a_{41}, a_{23}, a_{32} > 0$, $a_{34}, a_{43} < 0$, $a_{13} = a_{31} = a_{24} = a_{42} = 0$, and diagonal entries are arbitrary. The corresponding graph $H(A)$ is given in Fig. 1.

The only loop $(1, 2, 3, 4)$ is negative in both graphs $H(A)$ and $H(-A)$.

That is why, for any 4×4 diagonal matrix B, system (8.7) has no invariant orthants. But the question of global controllability of this system (if it has a full rank) seems to be open. These statements remain stable under small perturbations of the matrix A.

8.6. Classical Lie groups.

8.6.1. Complex simple Lie algebras. Let $\mathrm{M}(n;\mathbb{C})$ denote the set of all $n \times n$ complex matrices, and let $\mathfrak{d}(n;\mathbb{C})$ be the subset of all diagonal matrices in $\mathrm{M}(n;\mathbb{C})$.

The Lie algebra $\mathfrak{gl}(n;\mathbb{C})$ is the vector space $\mathrm{M}(n;\mathbb{C})$ endowed with the matrix commutator $[A, B] = AB - BA$ as a Lie bracket.

The Lie algebra $\mathfrak{sl}(n;\mathbb{C})$ is a subalgebra of $\mathfrak{gl}(n;\mathbb{C})$ consisting of all matrices with zero trace:

$$\mathfrak{sl}(n;\mathbb{C}) = \{A \in \mathfrak{gl}(n;\mathbb{C}) \mid \mathrm{tr}A = 0\}.$$

The subalgebra

$$\mathfrak{sl}(n;\mathbb{C}) \cap \mathfrak{d}(n;\mathbb{C}) = \{\mathrm{diag}(a_1, \dots, a_n) \mid a_1 + \dots + a_n = 0\}$$

is a Cartan subalgebra of $\mathfrak{sl}(n;\mathbb{C})$. The algebra $\mathfrak{sl}(n;\mathbb{C})$ is called an algebra of the type A_l, $l = n-1$.

Let $(\cdot,\cdot)$ be a nonsingular symmetric bilinear form on $\mathbb{C}^n$. The Lie algebra $\mathfrak{so}(n;\mathbb{C})$ consists of all endomorphisms A of $\mathbb{C}^n$ such that

$$(Ax, y) + (x, Ay) = 0, \qquad x, y \in \mathbb{C}^n.$$

Assume that the form $(\cdot,\cdot)$ is defined by the $n \times n$ matrix I given by

$$\begin{pmatrix} 0 & \mathrm{Id}_l \\ \mathrm{Id}_l & 0 \end{pmatrix} \quad \text{if } n = 2l \quad \text{or} \quad \begin{pmatrix} 0 & \mathrm{Id}_l & 0 \\ \mathrm{Id}_l & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } n = 2l+1,$$

where Id_l is the identity $l \times l$ matrix. Then

$$\mathfrak{so}(n;\mathbb{C}) = \{A \in \mathfrak{gl}(n;\mathbb{C}) \mid IA^{\mathrm{T}} + AI = 0\}.$$

In this case, the Cartan subalgebra can be chosen as a subspace of $\mathfrak{so}(n;\mathbb{C})$ consisting of matrices of the form

$$\begin{aligned} &\mathrm{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l) \quad \text{if } n = 2l, \\ &\mathrm{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l, 0) \quad \text{if } n = 2l+1. \end{aligned}$$

The algebra $\mathfrak{so}(n;\mathbb{C})$ is called an algebra of the type B_l for $n = 2l+1$ and an algebra of the type D_l for $n = 2l$.

Now let $(\cdot,\cdot)$ be a nonsingular skew-symmetric bilinear form on $\mathbb{C}^n$. The Lie algebra $\mathfrak{sp}(n;\mathbb{C})$ is defined as the set of all endomorphisms A of $\mathbb{C}^n$ such that

$$(Ax, y) + (x, Ay) = 0, \qquad x, y \in \mathbb{C}^n.$$

If the form $(\cdot,\cdot)$ is defined by the matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id}_l \\ -\mathrm{Id}_l & 0 \end{pmatrix},$$

then

$$\mathfrak{sp}(n;\mathbb{C}) = \{A \in \mathfrak{gl}(n;\mathbb{C}) \mid JA^{\mathrm{T}} + AJ = 0\}.$$

The set of all matrices in $\mathfrak{sp}(n;\mathbb{C})$ of the form

$$\mathrm{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l)$$

is a Cartan subalgebra of $\mathfrak{sp}(n;\mathbb{C})$. The algebra $\mathfrak{sp}(n;\mathbb{C})$ is called an algebra of the type C_n.

In Secs. 8.6.2, 8.6.3, and 8.6.4, we assume that Cartan subalgebras L_0 in Lie algebras A_l, B_l, C_l, and D_l are chosen as above. In particular, the condition $B \in L(0)$ will imply that the matrix B is diagonal.

The algebras A_l, B_l, C_l, and D_l are known as *classical complex Lie algebras.* In addition to them, there exist five exceptional Lie algebras denoted by G_2, F_4, E_6, E_7, and E_8.

The classical classification result on complex simple Lie algebras states that all Lie algebras

$$A_l,\ l \geq 1, \quad B_l,\ l \geq 2, \quad C_l,\ l \geq 3, \quad D_l,\ l \geq 4,$$

and

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

are simple, and any simple Lie algebra over $\mathbb{C}$ is isomorphic to exactly one of these.

8.6.2. Generation of classical Lie algebras. The central role for the controllability results in this subsection is played by the following proposition, which describes pairs of elements that generate classical Lie algebras.

Theorem 8.9. *Let L be the normal real form of a complex Lie algebra of type A_l, B_l, C_l, or D_l. Let elements $A, B \in L$ be such that $B \in L(0)$ is A-strongly regular. Then* $\mathrm{Lie}(A, B) = L$ *if and only if the matrix A is permutation-irreducible.*

Remark. The nontrivial part in this theorem is the sufficiency: if B is diagonal and A permutation-reducible, then it is easy to see that $\mathrm{Lie}(A, B)$ consists of permutation-reducible matrices.

8.6.3. Homogeneous systems.

Theorem 8.10. *Let G be a connected Lie group with a Lie algebra L that is a normal real form of a complex Lie algebra of type A_l, B_l, C_l, or D_l. Let elements $A, B \in L$ be such that $B \in L(0)$ is A-strongly regular. Then the system $\Gamma = \mathbb{R}A + \mathbb{R}B$ is controllable on G if and only if the matrix A is permutation-irreducible.*

Proof. For homogeneous systems, the controllability is equivalent to the rank condition. Thus, the statement follows from Theorem 8.9.

8.6.4. Nonhomogeneous systems.

Theorem 8.11. *Let G be a connected Lie group with a Lie algebra L that is a normal real form of a complex Lie algebra $\mathcal{L}$ of type A_l or D_l. Let elements $A, B \in L$ be such that*

(i) $B \in L(0)$ *is A-strongly regular*;

(ii) *let* $A = A(0) + \sum\{A(\alpha) \mid \alpha \in R\}$ *be the decomposition of* A *along the root spaces of* $\mathcal{L}$ *relative to* L_0*; see* (8.2)*. Then* $\mathrm{Kil}(A(s), A(-s)) < 0$ *for the maximal root* s*.*

Then the system $\Gamma = A + \mathbb{R}B$ *is controllable on* G *if and only if the matrix* A *is permutation-irreducible.*

This theorem is proved by an argument analogous to that used in the proof of Theorem 8.5 in Sec. 8.4.2. The only essential distinction is that the rank condition follows from Theorem 8.9.

8.7. Remarks. The controllability results of Sec. 8.4 were obtained by El Assoudi, Gauthier, and Kupka [10]. They are a culmination of the series of papers on controllability on semisimple Lie groups by Jurdjevic and Kupka [80, 81], Gauthier and Bornard [48], Gauthier, Kupka and Sallet [49], El Assoudi and Gauthier [8, 9], Silva Leite and Crouch [136], El Assoudi [7]. In particular, Theorem 8.7 was obtained by Gauthier and Bornard [48]. This paper also contains an easy procedure for verification whether a square matrix is permutation-irreducible.

Proposition 8.1 is due to Joseph [73].

Conjecture 8.1 on the noncontrollability of a single-input bilinear system with symmetric matrices in the right-hand side was suggested by Jurdjevic and Kupka [80].

Invariant orthants of bilinear systems were described by Sachkov [121] via application of bichromatic graphs to the study of invariant sets of dynamical systems; this idea is due to Hirsch [60].

Results of Sec. 8.6 are due to Silva Leite and Crouch [136]. In addition to Theorem 8.9, there are other results on generation of classical Lie algebras and Lie groups; see Crouch and Silva Leite [43], Silva Leite [132, 133, 134, 135], Albuquerque and Silva Leite [5].

Results related to subsemigroups of semisimple Lie groups can be found in papers by San Martin [129] and San Martin and Tonelli [130].

9. Nilpotent Lie Groups

A Lie algebra L is called *nilpotent* if its descending central series

$$L_{(1)} = [L, L],\ L_{(2)} = [L, L_{(1)}],\ \ldots,\ L_{(i)} = [L, L_{(i-1)}],\ \ldots,\quad i \in \mathbb{N},$$

stabilizes at zero, that is,

$$L \supset L_{(1)} \supset L_{(2)} \supset \cdots \supset L_{(N)} = \{0\}$$

for some $N \in \mathbb{N}$. Any nilpotent Lie algebra is solvable, since $L_{(i)} \supset L^{(i)}$, $i \in \mathbb{N}$, where $L^{(i)}$ denotes an element of the derived series

$$L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}],\ \ldots,\ L^{(i)} = [L^{(i-1)}, L^{(i-1)}],\ \ldots,\quad i \in \mathbb{N}.$$

Another equivalent characterization of nilpotency of L is that all adjoint operators $\operatorname{ad} x$, $x \in L$, are nilpotent, and thus, have zero spectrum.

9.1. Arbitrary systems. Controllability of a right-invariant system $\Gamma \subset L$ on a nilpotent Lie group G is completely characterized in terms of the wedge, i.e., the topologically closed convex positive cone generated by Γ:

$$W(\Gamma) = \operatorname{cl}(\operatorname{co}(\Gamma)) \subset L.$$

Since $\Gamma \subset W(\Gamma) \subset \operatorname{LS}(\Gamma)$, it is obvious that Γ and $W(\Gamma)$ are controllable or noncontrollable, simultaneously.

Theorem 9.1. *Let G be a nilpotent connected Lie group with a Lie algebra L, and let $\Gamma \subset L$ be a right-invariant system on G that generates L as a Lie algebra. Then Γ is controllable on G if and only if one of the following conditions holds*:

(i) $(\operatorname{int}_{W-W} W) \cap L^{(1)} \neq \varnothing$ *or*

(ii) $\operatorname{int} \operatorname{cl}(L^{(1)} + W) \cap \exp^{-1}(e) \neq \varnothing$,

where $W = W(\Gamma)$ is the wedge generated by Γ.

Remark. Here $\operatorname{int}_{W-W} W$ is the interior of the wedge W relative to the vector space $W - W$ generated by W, and e is the identity of the Lie group G.

The sufficiency in Theorem 9.1 follows from the description of maximal open subsemigroups S of nilpotent Lie groups in terms of their tangent objects:

$$L(S) = \{\, x \in L \mid \exp(\mathbb{R}_+ x) \subset \operatorname{cl}(S)\,\}.$$

An open subsemigroup S of a Lie group G is proper, i.e., $S \neq G$ if and only if $e \notin S$. Hence the set of open subsemigroups in G is inductive, and any proper subsemigroup is contained in a *maximal* one.

Theorem 9.2. *Let G be a nilpotent connected Lie group, and let S be a maximal open proper subsemigroup of G. Then $L(S)$ is a half-space bounded by a codimension one subalgebra in L.*

For necessity in Theorem 9.1, the Hahn–Banach theorem yields that $W + L^{(1)}$ is contained in a half-space in L; then $\exp\left(\operatorname{int}\left(W + L^{(1)}\right)\right)$ is a proper open semigroup of G; this implies that $\exp(W)$ is contained in a proper subsemigroup of G.

The controllability test of Theorem 9.1 is essentially nilpotent. This result is not true for the group $\operatorname{SL}(2; \mathbb{R})$. It also fails in the following solvable non-nilpotent example.

Example 9.1. Let G be a (unique) two-dimensional connected simply connected non-Abelian Lie group, which is represented by matrices as follows:

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \in \mathbb{R} \right\}.$$

The Lie group G is solvable but is not nilpotent. Its Lie algebra has the form

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Consider the following wedge in L:

$$W = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R},\ b \geq 0 \right\}.$$

A direct computation shows that

$$\exp(\mathbb{R}_+ W) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \geq 0 \right\},$$

which is a proper subsemigroup of G; thus, W is not controllable on G. On the other hand, it is easy to see that both conditions (i) and (ii) of Theorem 9.1 hold for the wedge W in this example.

9.2. Abelian groups. Let G be a (connected) *Abelian* Lie group. Then $G = \mathbb{R}^{n-k} \times T^k$ for some $k \leq n$, where $n = \dim G$ and $T^k = S^1 \times \cdots \times S^1$ is the k-dimensional torus.

For such Lie groups, Theorem 9.1 implies the following.

Corollary 9.1. *Let G be an Abelian connected Lie group with a Lie algebra L, and let $\Gamma \subset L$ be a right-invariant system on G. Then Γ is controllable on G if and only if*

$$\operatorname{int}(\operatorname{cl}(\operatorname{co}(\Gamma))) \cap \exp^{-1}(e) \neq \varnothing.$$

If, in addition, G is simply connected, then Γ is controllable if and only if

$$\operatorname{int}(\operatorname{cl}(\operatorname{co}(\Gamma))) \ni e.$$

9.3. Quotient systems. Let G be an arbitrary Lie group with a Lie algebra L. Let h be an ideal of L, and let H be the corresponding connected subgroup of G. Assume that H is closed, and so G/H is a Lie group. Denote the projection from G onto G/H by π and its differential by π_*. The projection of the system Γ onto G/H is well defined:

$$\pi_*(\Gamma) = \{ \pi_* v \mid v \in \Gamma \} \subset L/h.$$

Notice that the controllability of the system Γ on G implies the controllability of its projection $\pi_*(\Gamma)$ on G/H.

The derived subalgebra $L^{(1)}$ is an ideal of L, and for simply connected G, its derived subgroup $G^{(1)} = [G, G]$ is closed. Moreover, the quotient group $G/G^{(1)}$ is Abelian. Thus, the above construction and Corollary 9.1 allow us to give the following general necessary controllability condition for control affine systems

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \ldots, m \right\} \subset L \tag{9.1}$$

on simply-connected Lie groups.

Introduce the following notation for the Lie algebra generated by the vector fields in Γ near controls:

$$L_0 = \mathrm{Lie}(B_1, \ldots, B_m),$$

Theorem 9.3. *Let a connected Lie group G be simply connected. If a control affine right-invariant system* (9.1) *is controllable, then*

(1) $\pi_*(L_0) = L/L^{(1)}$;

(2) $m \geq \dim L - \dim L^{(1)}$.

Proof. (1) If Γ is controllable on G, then $\pi_*(\Gamma)$ is controllable on the Abelian simply-connected Lie group $G/G^{(1)}$. Then it follows from Corollary 9.1 that $\pi_*(L_0) = L/L^{(1)}$.

(2) The Lie algebra $\pi_*(L_0)$ is Abelian and is spanned by the vectors $\pi_* B_1$, …, $\pi_* B_m$. Therefore,

$$m \geq \dim(\pi_*(L_0)) = \dim(L/L^{(1)}) = \dim L - \dim L^{(1)}.$$

Remark. This theorem implies that right-invariant systems on a simply-connected Lie group G with nontrivial $G/G^{(1)}$ essentially differ from right-invariant systems on semisimple Lie groups (notice that if G is semisimple, then $G^{(1)} = G$). In semisimple Lie groups, $m = 2$ is sufficient for controllability of a generic control affine right-invariant system; see Theorem 8.3. But Theorem 9.3 yields a lower bound

$$m \geq \dim G/G^{(1)}$$

for the number of controlled vector fields $B_1, \ldots, B_m$ that is necessary to achieve the controllability on the simply-connected G.

9.4. Control-affine systems. For control-affine right-invariant systems (9.1) on simply-connected nilpotent Lie groups, there is a simple controllability criterion in terms of the Lie subalgebra L_0.

Theorem 9.4. *Let G be a connected, simply-connected nilpotent Lie group. Then system* (9.1) *is controllable on G if and only if $L_0 = L$.*

The sufficiency of the condition $L_0 = L$ for controllability of Γ is valid for arbitrary Lie groups G: it follows from the inclusion $L_0 \subset \mathrm{LS}(\Gamma)$. So, the essential part is the necessity. Here a key role is played by the necessary controllability conditions in terms of the notion of a *symplectic vector*.

Consider the co-adjoint representation ρ^* of the group G in the dual space L^* of L. For any covector $\lambda \in L^*$, the co-adjoint orbit θ_λ of λ by the ρ^* action $\theta_\lambda = \rho^*_G(\lambda)$ is a smooth submanifold of L^* diffeomorphic to the homogeneous space G/E_λ, where E_λ is the isotropy subgroup of λ, $E_\lambda = \{\, g \in G \mid \rho_g^*(\lambda) = \lambda \,\}$. Further, the system Γ can be projected from G onto the homogeneous space $G/E_\lambda \simeq \theta_\lambda$, and the controllability of Γ on G obviously implies the controllability of its projection Γ_λ on G/E_λ. This leads us to necessary controllability conditions in terms of the co-adjoint representation.

Definition 9.1. A covector $\lambda \in L^*$ is called a *symplectic vector* for $w \in L$ if the co-adjoint orbit θ_λ is not trivial and $\langle w, \beta \rangle > 0$ for all $\beta \in \theta_\lambda$.

(We denote by $\langle \cdot, \cdot \rangle$ the pairing of a vector and covector.)

Theorem 9.5. *If there is a vector field $\xi \in L$ belonging to the centralizer of the subalgebra L_0 such that the nonzero vector field $[A, \xi]$ has a symplectic vector, then system* (9.1) *cannot be controllable on G.*

In fact, the existence of such vector field $\xi \in L$ yields that the function

$$f_\xi \;:\; \theta_\lambda \to \mathbb{R}, \qquad \beta \mapsto f_\xi(\beta) = -\langle \xi, \beta \rangle$$

is strictly increasing on trajectories of the projection of Γ onto the co-adjoint orbit θ_λ. Indeed, the solution to the Cauchy problem $\dot{g}(t) = A(g(t))$, $g(0) = g_0$, is given by $g(t) = \exp(tA)g_0$. Further, the function

$$h(t) = \operatorname{Ad}\left(g(t)^{-1}\right) = \operatorname{Ad}\left(g_0^{-1}\right) \circ \exp(-t \operatorname{ad} A)$$

has the derivative

$$\dot{h}(t) = \operatorname{Ad}\left(g_0^{-1}\right) \circ \exp(-t \operatorname{ad} A) \circ (-\operatorname{ad} A) = -\operatorname{Ad}\left(g(t)^{-1}\right) \circ \operatorname{ad} A.$$

Now, for $\lambda \in L^*$, the co-adjoint action ρ^* of the element $g \in G$ is determined by

$$\rho_g^*(\lambda) = \operatorname{Ad}^*\left(g^{-1}\right)\lambda;$$

consequently, for any $\xi \in L$ and $\lambda \in L^*$,

$$\begin{aligned}
\frac{d}{dt} f_\xi(\rho_{g(t)}^*(\lambda)) \;&=\; -\frac{d}{dt}\langle \xi, \rho_{g(t)}^*(\lambda)\rangle = -\frac{d}{dt}\langle \xi, \operatorname{Ad}^*\left(g(t)^{-1}\right)\lambda\rangle \\
&=\; -\frac{d}{dt}\langle \operatorname{Ad}\left(g(t)^{-1}\right)\xi, \lambda\rangle = \langle \operatorname{Ad}\left(g(t)^{-1}\right) \circ \operatorname{ad} A(\xi), \lambda\rangle \\
&=\; \langle [A, \xi], \rho_{g(t)}^*(\lambda)\rangle.
\end{aligned}$$

Thus, if λ is a symplectic vector for $[A, \xi]$, then f_ξ increases along co-adjoint orbits of trajectories of the field A. If, in addition, $\operatorname{ad}\xi$ vanishes on the subalgebra L_0, then the same holds for trajectories of the whole system Γ; this is impossible for a controllable system.

Another important fact for necessity in Theorem 9.4 is the following proposition related to *hypersurface* systems, i.e., control-affine systems (9.1) with L_0 being a codimension one subalgebra of L.

Denote by G_0 the connected subgroup of the Lie group G corresponding to the subalgebra L_0.

Theorem 9.6. *Let $\Gamma \subset L$ be a control-affine system* (9.1) *on a connected Lie group G such that L_0 is a codimension one* ideal *of L. Then*

1. *If G_0 is closed in G, then Γ is controllable if and only if $A \notin L_0$ and $G/G_0 \simeq S^1$.*
2. *If G_0 is not closed in G, then Γ is controllable if and only if $A \notin L_0$.*

Remark. The previous theorem holds without assumption that L_0 is an ideal; this is important for a generalization of Theorem 9.4 to a subclass of solvable Lie groups including nilpotent ones (see Sec. 13 below).

Now we outline the scheme of proof of the necessity in Theorem 9.4. Assume that the system Γ is controllable on the group G. Then the theory of symplectic vectors implies that the subalgebra L_0 is an ideal of L. The rank condition for Γ holds: $\text{Lie}(\Gamma) = \text{Lie}(A, L_0) = L$; hence, L_0 has codimension 0 or 1 in L. But the codimension one case is impossible, since then Theorem 9.6 yields $G/G_0 \simeq S^1$; this contradicts the simple connectedness of G. Thus, $L_0 = L$, and the necessity in Theorem 9.4 follows.

Example 9.2. Let G be the Heisenberg group of dimension $2p+1$. It can be represented as a subgroup of $\text{GL}(p+2;\mathbb{R})$ generated by the matrices

$$\text{Id} + X_i, \ \text{Id} + Y_i, \ Z, \quad i = 1, \dots, p,$$

where

$$X_i = E_{1,i+1}, \ Y_i = E_{i+1,p+2}, \quad i = 1, \dots, p.$$

The Lie algebra L of G is spanned by the matrices

$$X_i, \ Y_i, \ Z, \quad i = 1, \dots, p,$$

with the nonzero brackets

$$[X_i, Y_i] = Z, \quad i = 1, \dots, p.$$

The Heisenberg group G is simply-connected and nilpotent; hence, Theorem 9.4 describes all controllable systems on G.

9.5. Remarks. The controllability test for arbitrary right-invariant systems and the description of maximal subsemigroups in nilpotent Lie groups (Sec. 9.1) is due to Hilgert, Hofmann, and Lawson [57].

The result on quotient systems (Sec. 9.3) was obtained by Sachkov [118].

The criterion for control affine systems (Sec. 9.4) was given by Ayala [12], and the notion of a symplectic vector is due to Ayala and Vergara [13].

The controllability of projections of right-invariant systems onto nilpotent and solvable manifolds might be studied via application of the theory of flows on these manifolds, see e.g., the book by Auslender, Green, and Hahn [11]. This can be important for studying the local controllability of nonlinear systems via nilpotent approximations (Crouch and Byrnes [44]).

10. Products of Lie Groups

In this section, we present controllability conditions on products of vector groups with nilpotent Lie groups. The results obtained for such Lie groups can be viewed as a generalization of results for nilpotent Lie groups; see Sec. 9.1.

Theorem 10.1. *Let G be a connected Lie group, C be a connected compact subgroup of G, and let N be a nilpotent normal subgroup of G such that $G = C \cdot \mathrm{cl}(N)$. If W is a wedge in L that generates L as a Lie algebra, then W is controllable if and only if*

$$\mathrm{int}_{W-W}(W) \cap (L(C) + L^{(1)}) \neq \varnothing.$$

The previous controllability result is proved via the following description of all maximal open semigroups in products of compact and nilpotent groups.

Theorem 10.2. *Let G be a connected Lie group, C be a connected compact subgroup of G, and let N be a nilpotent normal subgroup of G such that $G = C \cdot \mathrm{cl}(N)$. If S is a maximal open subsemigroup of G, then its tangent wedge*

$$L(S) = \{X \in L \mid \exp(tX) \in \mathrm{cl}(S)\ \forall t \geq 0\}$$

is a half-space bounded by an ideal in L.

10.1. Remarks. The results in this section are due to Hilgert [55].

Another important (and more general) result on maximal semigroups related to controllability is the characterization of maximal subsemigroups in Lie groups with cocompact radical by Lawson; see Sec. 11.

11. Lie Groups with Cocompact Radical

Denote by $\mathrm{Rad}\, G$ the *radical* of a Lie group G, i.e., the maximal solvable normal subgroup of G. In this section, we assume that a Lie group G has *cocompact radical*, that is, the quotient group $K = G/\mathrm{Rad}\, G$ is compact. This wide class of Lie groups contains

(i) solvable Lie groups ($K = \{e\}$);

(ii) compact Lie groups;

(iii) semidirect products of a vector space V with a compact Lie group ($V \subset \mathrm{Rad}\, G$).

11.1. Controllability conditions and maximal subsemigroups. The next theorem gives a Lie-algebraic description of controllability on Lie groups with cocompact radical, which is complete in the simply-connected case.

Theorem 11.1. *Assume that $G/\operatorname{Rad} G$ is compact; let $\Gamma \subset L$ be a right-invariant system that satisfies the rank condition $\operatorname{Lie}(\Gamma) = L$. If Γ is not contained in any half-space of L with boundary being a subalgebra, then Γ is controllable on the connected Lie group G. The converse holds if G is simply-connected.*

This result is a consequence of the following classification of maximal subsemigroups of Lie groups with cocompact radical.

Theorem 11.2. *The maximal subsemigroups M with nonempty interior of a connected, simply-connected Lie group G with compact $G/\operatorname{Rad} G$ are in a one-to-one correspondence with their tangent objects*

$$L(M) = \{A \in L \mid \exp(tA) \in \operatorname{cl}(M)\ \forall t \geq 0\},$$

and the latter are exactly the closed half-spaces with boundary being a subalgebra. Furthermore, M is the semigroup generated by $\exp(L(M))$.

Theorem 11.1 follows from Theorem 11.2, since the attainable set of any non-controllable right-invariant system $\Gamma \subset L$, $\operatorname{Lie}(\Gamma) = L$, is a proper subsemigroup of G contained in some maximal subsemigroup with non-empty interior.

11.2. Reductive Lie groups. A Lie algebra L is called *reductive* if its radical, i.e., the maximal solvable ideal, coincides with its center. L is reductive if and only if the derived subalgebra $L^{(1)}$ is semisimple. In this case, L is the direct sum of its center and $L^{(1)}$. A Lie group is called *reductive* if it has reductive Lie algebra.

In this subsection, we present a characterization of controllable systems Γ on a reductive group G under the assumptions that Γ is a closed convex cone in L, Γ is pointed, i.e., it has the zero edge: $\Gamma \cap -\Gamma = \{0\}$, and is invariant under the adjoint action of the group K, where NAK is an Iwasawa decomposition of G.

Recall that (see Sec. 4)

(1) a right-invariant system Γ is controllable if and only if the closed convex cone $\operatorname{cl}(\operatorname{co}(\Gamma))$ generated by Γ is controllable;

(2) a system Γ is controllable, i.e., $\mathbb{A} = G$ if and only if the closure $\operatorname{cl}(\mathbb{A})$ coincides with G, provided that Γ satisfies the rank condition $\operatorname{Lie}(\Gamma) = L$.

Let $G = NAK$ be an Iwasawa decomposition of a Lie group G, where N, A, and K are respectively a maximal nilpotent subgroup, a principal vector subgroup, and a maximal compact subgroup of G. Denote by $L(N)$ and $L(K)$ the Lie algebras of the Lie groups N and K, respectively, and by $L^{(1)}(K)$ the derived subalgebra of K.

Theorem 11.3. *Let G be a connected, simply-connected reductive Lie group with Iwasawa decomposition NAK. Let Γ be an $\operatorname{Ad}(K)$-invariant convex cone in L satisfying $\Gamma \cap -\Gamma = \{0\}$ and $\operatorname{int}\Gamma \neq \varnothing$. Then the following assertions hold.*

(i) *If $(\operatorname{int}\Gamma) \cap (L(N) + L^{(1)}(K)) \neq \varnothing$, then Γ is controllable.*

(ii) *If* $\Gamma \cap (L(N) + L^{(1)}(K)) = \{0\}$, *then*

$$\Gamma = \mathrm{LS}(\Gamma) = \{X \in L \mid \exp(\mathbb{R}_+ X) \subset \mathrm{cl}(\mathbb{A})\}$$

and Γ *is not controllable.*

(iii) *If* $\varnothing \neq (\Gamma \cap (L(N) + L^{(1)}(K))) \setminus \{0\} \subset \partial\Gamma$, *then* Γ *is not controllable.*

The main idea of the proof of this theorem is to describe a reductive group as a homogeneous space of a group with cocompact radical, and after that, use the characterization of maximal subsemigroups in such groups given by Theorem 11.2.

11.3. Remarks. The description of maximal subsemigroups in Lie groups with cocompact radical and conditions of controllability on such Lie groups in Sec. 11.1 are due to Lawson [94].

The controllability result for reductive Lie groups in Sec. 11.2 was obtained by Hilgert [56].

11.3.1. The rank condition and hypersurface principle. The customary procedure for verifying the noncontrollability is either to show the violation of the rank controllability condition; see Theorem 2.3, or to construct a (not necessarily smooth) hypersurface in the state space of a system intersected by all trajectories of the system in one direction only; see e.g., the hypersurface principle in Theorem 12.2. By Lawson's Theorem 11.1, for right-invariant systems on simply-connected Lie groups with cocompact radical such hypersurface can always be found among codimension one subgroups. An interesting question is whether any full-rank noncontrollable right-invariant system have such codimension one subgroup? A positive answer will give a new method for obtaining sufficient controllability conditions, and a negative one will give an example of a complex obstruction to controllability.

12. Hypersurface Systems

The class of control systems in $\mathbb{R}^n$ with $(n-1)$ independent controls has specific features that simplify their study, especially in the case of unbounded controls. The more so this is true for right-invariant systems.

Definition 12.1. A control affine right-invariant system

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \dots, m \right\} \subset L \tag{12.1}$$

is called *hypersurface* if the Lie algebra L_0 generated by the vector fields $B_1, \dots, B_m$ is a codimension one subalgebra of L:

$$\dim L_0 = \dim \mathrm{Lie}(B_1, \dots, B_m) = \dim L - 1.$$

Denote by G_0 the connected subgroup of G corresponding to the subalgebra L_0.

The controllability of hypersurface right-invariant systems is completely characterized by the following proposition.

Theorem 12.1. *Let Γ be a hypersurface control affine system* (12.1) *on a connected Lie group G. Then*

(1) *If G_0 is closed in G, then Γ is controllable if and only if $A \notin L_0$ and $G/G_0 \simeq S^1$;*

(2) *If G_0 is not closed in G, then Γ is controllable if and only if $A \notin L_0$.*

Proof. The condition $A \notin L_0$ is necessary for controllability in both cases (1) and (2), since it is equivalent to the rank condition $\mathrm{Lie}(\Gamma) = \mathrm{Lie}(A, L_0) = L$. Also, we note that $L_0 \subset \mathrm{LS}(\Gamma)$; that is why Γ is controllable if and only if the extended system $\tilde{\Gamma} = \mathrm{cl}(\mathrm{co}(\Gamma)) = \mathbb{R}_+ A + L_0$ is controllable.

(1) If $\mathrm{cl}(G_0) = G_0$, then the right coset space G/G_0 is a smooth one-dimensional manifold, i.e., the line $\mathbb{R}$ or the circle S^1. Since any point of the right coset $G_0 x$ is reachable from x for the system $\tilde{\Gamma}$ for any $x \in G$, we can project $\tilde{\Gamma}$ onto G/G_0. It is easy to see that the projected system is controllable if $G/G_0 = S^1$ and noncontrollable if $G/G_0 = \mathbb{R}$.

(2) If the codimension one subgroup G_0 is not closed in G, then it is dense in G; thus, the reachable set $\mathbb{A}$ is also dense in G. If $A \notin L_0$, then the system Γ has a full rank; thus, it is controllable by Theorem 2.8.

Remark. Theorem 12.1 generalizes the analogous criterion of Theorem 9.6 with an additional assumption that L_0 is an ideal of L.

Corollary 12.1. *A hypersurface system cannot be controllable on a simply-connected Lie group.*

Proof. If G is simply-connected, then its codimension one subgroup G_0 is closed. Furthermore, G is simply-connected; that is why G/G_0 is simply-connected as well. Thus, $G/G_0 = \mathbb{R}$, and it follows from Theorem 12.1 that Γ is not controllable.

The previous propositions imply the following *hypersurface principle*, the general necessary controllability condition for simply-connected Lie groups.

Theorem 12.2. *Let $\Gamma \subset L$ be a control affine system* (12.1) *on a connected, simply-connected Lie group G. Assume that there exists a codimension one subalgebra l of the Lie algebra L containing L_0. Then Γ is not controllable.*

Proof. The system Γ can be extended to an affine system of the form

$$\Gamma_1 = \left\{ A + \sum_{i=1}^{m} u_i B_i + \sum_{i=m+1}^{k} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \ldots, k \right\},$$

where $B_{m+1}, \dots, B_k$ complement $B_1, \dots, B_m$ to a basis of the subalgebra l. By Corollary 12.1, the system Γ_1 is not controllable, and therefore, Γ is not controllable too.

The sense of this proposition is that if the codimension one subalgebra $l \supset L_0$ exists, then attainable set of Γ lies "to one side" of the connected codimension one subgroup of G corresponding to l: by the simple connectedness of G, this codimension one subgroup separates G into two disjoint parts.

12.1. Remarks. General hypersurface nonlinear systems were studied by Hunt [67, 68].

The results of this section are due to Sachkov [118].

The hypersurface principle given by Theorem 12.2, is a necessary controllability condition for an arbitrary simply-connected Lie group. By Lawson's Theorem 11.1, if a simply-connected Lie group has a cocompact radical, then this principle is also sufficient for controllability. It would be interesting to extend the class of simply-connected Lie groups with cocompact radical so that the hypersurface principle remain to be a controllability criterion.

13. Completely Solvable Lie Groups

In this section, we assume that Γ is a control-affine system (12.1) and give controllability conditions for a subclass of the class solvable Lie groups.

Definition 13.1. A solvable Lie algebra L is called a *completely solvable* if all adjoint operators $\operatorname{ad} x$, $x \in L$, have real spectra. A Lie group is *completely solvable* if it has a completely solvable Lie algebra.

The triangular group $\mathrm{T}(n; \mathbb{R})$ (see Example 13.1 below) is completely solvable as well as any of its subgroups. Nilpotent Lie groups are completely solvable, since adjoint operators in nilpotent Lie algebras have zero spectrum. On the other hand, the group of motions of the plane $\mathrm{E}(2; \mathbb{R})$ is, e.g., solvable but not completely solvable (the group $\mathrm{E}(2; \mathbb{R})$, and its simply-connected covering $\tilde{\mathrm{E}}(2; \mathbb{R})$ are treated in Sec. 15).

Completely solvable Lie algebras have many codimension one subalgebras (this is crucial for the controllability test for completely solvable Lie groups).

Lemma 13.1. *If L is a completely solvable Lie algebra, then, for any subalgebra $l_1 \subset L$, $l_1 \neq L$, there exists a subalgebra $l_2 \subset L$ such that $l_1 \subset l_2$ and $\dim l_2 = \dim l_1 + 1$.*

It turns out that the controllability criterion for systems affine in control on nilpotent Lie groups (Theorem 9.4) is valid for completely solvable Lie groups as well.

Theorem 13.1. *Let G be a connected, simply-connected completely solvable Lie group. Then system (12.1) is controllable on G if and only if $L_0 = L$.*

Proof. *Sufficiency.* If $L_0 = L$, then $\mathrm{LS}(\Gamma) \supset \mathrm{LS}(L_0) = L$. By Theorem 4.3, the system Γ is controllable.

The *necessity* follows from Theorem 12.2 and Lemma 13.1.

Example 13.1. Let $G = \mathrm{T}(n;\mathbb{R})$ be the group of all real $n \times n$ upper triangular matrices with positive diagonal entries. We see that $\mathrm{T}(n;\mathbb{R})$ is a connected, simply-connected, and completely solvable Lie group. Its Lie algebra $L = \mathfrak{t}(n;\mathbb{R})$ consists of all $n \times n$ upper triangular matrices. The derived subalgebra $L^{(1)}$ consists of all strictly upper triangular matrices, and $L/L^{(1)}$ is the n-dimensional Abelian Lie algebra of all diagonal $n \times n$ matrices.

By Theorem 13.1, a control affine system Γ is controllable on $\mathrm{T}(n;\mathbb{R})$ if and only if $L_0 = L$.

By Theorem 9.3, the controllability of Γ on $\mathrm{T}(n;\mathbb{R})$ can be attained with not less than $n = \dim L/L^{(1)}$ controlled vector fields. This lower estimate is sharp. For example, the system $\Gamma = \{\, A + \sum_{i=1}^{n} u_i B_i \mid u_i \in \mathbb{R} \,\}$ with $B_i = E_{ii} + E_{i,i+1}$ for $i = 1, \ldots, n-1$ and $B_n = E_{nn}$ is controllable on $\mathrm{T}(n;\mathbb{R})$. Indeed, it is easy to see that $\mathrm{Lie}(B_1, \ldots, B_n) = \mathfrak{t}(n;\mathbb{R})$.

Example 13.2. Let $G = \mathrm{E}(2;\mathbb{R})$ be the Euclidean group of motions of the two-dimensional plane $\mathbb{R}^2$. $\mathrm{E}(2;\mathbb{R})$ is a connected but not a simply-connected Lie group. It can be represented by 3×3 matrices of the form

$$\begin{pmatrix} c_{11} & c_{12} & b_1 \\ c_{21} & c_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = (c_{ij}) \in \mathrm{SO}(2;\mathbb{R}), \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2,$$

where C is a rotation matrix and b is a translation vector. The corresponding matrix Lie algebra $L = \mathfrak{e}(2;\mathbb{R})$ is spanned by the matrices $A_1 = E_{13}$, $A_2 = E_{23}$, and $A_3 = E_{21} - E_{12}$. We have $L^{(1)} = \mathrm{span}(A_1, A_2)$ and $L^{(2)} = \{0\}$; therefore, L is solvable.

Consider the right-invariant system $\Gamma = \{\, A_1 + uA_3 \mid u \in \mathbb{R} \,\}$. We use the Lie saturation technique and show that the system Γ is controllable on $\mathrm{E}(2;\mathbb{R})$.

We have $A_1, \pm A_3 \in \mathrm{LS}(\Gamma)$. That is why $\exp(s \operatorname{ad} A_3)A_1 \in \mathrm{LS}(\Gamma)$ for any $s \in \mathbb{R}$. But $\exp(s \operatorname{ad} A_3)A_1 = (\cos s)A_1 + (\sin s)A_2$. Consequently, $\mathrm{span}(A_1, A_2) \subset \mathrm{LS}(\Gamma)$; therefore, $\mathrm{LS}(\Gamma) = L$. Thus, Γ is controllable on $\mathrm{E}(2;\mathbb{R})$.

Obviously, Γ can also be considered as a right-invariant system on the simply-connected covering group $\tilde{\mathrm{E}}(2;\mathbb{R})$ of $\mathrm{E}(2;\mathbb{R})$. The above proof of controllability of Γ on $\mathrm{E}(2;\mathbb{R})$ is purely algebraic; i.e., it does not use any global geometric properties of $\mathrm{E}(2;\mathbb{R})$. That is why Γ is controllable on $\tilde{\mathrm{E}}(2;\mathbb{R})$ as well.

The spectrum of the operator $\operatorname{ad} A_3$ consists of $\pm i$ and 0. Therefore, this example shows us that the assumption on the complete solvability of L, i.e., of the realness of the spectrum of adjoint operators in Theorem 13.1 is essential. Detailed controllability conditions for right-invariant systems on the group $\mathrm{E}(2;\mathbb{R})$ and its simply-connected covering $\tilde{\mathrm{E}}(2;\mathbb{R})$ are given in Example 15.2.

13.1. Remarks. The results of this section were obtained by Sachkov [118].

Completely solvable Lie algebras and Lie groups are also called *triangular over* $\mathbb{R}$ or algebras (resp. groups) *of type* (R), see, e.g., the survey by Vinberg, Gorbatcevich, and Onishchik [148].

13.1.1. Lie algebras that are difficult to control. For any Lie group G and any system affine in control

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\}$$

on G, the controllability of the homogeneous part

$$\Gamma_0 = \left\{ \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\}$$

is sufficient for the controllability of Γ on G. We call a Lie algebra L *difficult to control* if any affine in control system $\Gamma \subset L$ and its homogeneous part Γ_0 are simultaneously controllable or noncontrollable (on the connected simply connected Lie group G corresponding to L). In Lie algebras L that are difficult to control, the drift term A in an affine system $\Gamma \subset L$ does not help in control, which is not the case for general Lie algebras.

There is an expanding chain of classes of Lie algebras that are difficult to control:

$$\text{Abelian} \subset \text{nilpotent} \subset \text{completely solvable}. \tag{13.1}$$

The Abelian case is Corollary 9.1, the nilpotent one is Theorem 9.4, and the completely solvable one is Theorem 13.1.

On the other hand, the Lie algebra of the group $\mathrm{E}(2;\mathbb{R})$ of motions of the plane is solvable, but not completely solvable and is not difficult to control; see Example 15.2.

By the hypersurface principle (Theorem 12.2), all Lie algebras satisfying the following property:

$$\left\{ \begin{array}{l} \text{any subalgebra } l \subset L,\ l \neq L, \text{ is contained} \\ \text{in a codimension one subalgebra of } L \end{array} \right\} \tag{13.2}$$

are difficult to control. The author does not know, whether this inclusion is strict. By Lemma 13.1, completely solvable Lie algebras satisfy property (13.2). The natural question is, whether there are Lie algebras difficult to control not contained in chain (13.1)? If yes, can this chain be continued by any reasonable class of Lie algebras? The theory on codimension one subalgebras of Lie algebras of Hofmann [62, 63, 65] can be important for this question.

13.1.2. Subalgebras of codimension one and two. The solution of the controllability problem for completely solvable Lie groups (see Sec. 13) is based on the following fact: any proper subalgebra of a real *completely solvable* Lie algebra is contained in a codimension one subalgebra. On the other hand, any proper subalgebra of a real *solvable* Lie algebra is included in some subalgebra of codimension one or two.

This suggests the following approach to the controllability on solvable Lie groups. Project a system along the connected subgroup corresponding to the indicated codimension one or two subalgebra. Then (1) if this group is closed and normal, we obtain a right-invariant system on a one- or two-dimensional Lie group (such systems are transparent); (2) if this subgroup is closed, we obtain a nonlinear system on a one- or two-dimensional smooth manifold (such systems are tractable by the nonlinear controllability theory); (3) and if this subgroup is not closed, then try to apply and develop the theory of control systems on foliations.

14. Lie Groups Differing from their Derived Subgroups

Lie algebras L that satisfy the condition $L \neq L^{(1)}$ form a wide class that contains the class of solvable Lie algebras but does not coincide with it; for example, $\mathfrak{gl}(n;\mathbb{R})$ is not solvable and has the derived subalgebra $\mathfrak{sl}(n;\mathbb{R})$. On the other hand, if a Lie algebra L is semisimple, then $L = L^{(1)}$. The converse is not true: the Lie algebra of infinitesimal motions of the three-dimensional space $\mathfrak{e}(2;\mathbb{R}) = \mathbb{R}^3 \leftthreetimes \mathfrak{so}(3;\mathbb{R})$ is not semisimple, although it coincides with its derived subalgebra.

In this section, we present controllability conditions for single-input systems

$$\Gamma = \{\, A + uB \mid u \in \mathbb{R} \,\} = A + \mathbb{R}B \subset L \tag{14.1}$$

on a Lie group G that does not coincide with its derived subgroup $G^{(1)}$. Consequently, we assume that $L \neq L^{(1)}$.

14.1. Notation and definitions. First, we introduce the notation connected with eigenvalues and eigenspaces of the adjoint operator $\operatorname{ad} B$ in L.
The derived subalgebra and the second derived subalgebra are

$$L^{(1)} = [L, L], \qquad L^{(2)} = [L^{(1)}, L^{(1)}];$$

the complexifications of L and $L^{(i)}$, $i = 1, 2$, are

$$L_c = L \otimes_{\mathbb{R}} \mathbb{C} \qquad L_c^{(i)} = L^{(i)} \otimes_{\mathbb{R}} \mathbb{C};$$

the adjoint representations and operators are

$$\operatorname{ad} : L \to \operatorname{End}(L), \qquad (\operatorname{ad} B)X = [B, X] \quad \forall X \in L,$$

$$\operatorname{ad}_c : L_c \to \operatorname{End}(L_c), \qquad (\operatorname{ad}_c B)X = [B, X] \quad \forall X \in L_c;$$

spectra of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$, are

$$\operatorname{Sp}^{(i)} = \left\{\, a \in \mathbb{C} \mid \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(i)}} - a \operatorname{Id}\right) \neq \{0\} \,\right\};$$

real and complex spectra of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$, are

$$\mathrm{Sp}_r^{(i)} = \mathrm{Sp}^{(i)} \cap \mathbb{R}, \qquad \mathrm{Sp}_c^{(i)} = \mathrm{Sp}^{(i)} \setminus \mathbb{R};$$

complex eigenspaces of $\operatorname{ad}_c B|_{L_c^{(1)}}$ are

$$L_c(a) = \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(1)}} - a \operatorname{Id}\right);$$

real invariant subspaces of $\operatorname{ad} B|_{L^{(1)}}$, which are one-dimensional for real $a \in \mathrm{Sp}_r^{(1)}$ and two-dimensional for complex $a \in \mathrm{Sp}_c^{(1)}$, are

$$L(a) = (L_c(a) + L_c(\overline{a})) \cap L;$$

complex root subspaces of $\operatorname{ad}_c B|_{L_c^{(i)}}$, $i = 1, 2$, are

$$L_c^{(i)}(a) = \bigcup_{N=1}^{\infty} \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(i)}} - a \operatorname{Id}\right)^N;$$

real invariant subspaces of $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$, real analogues of the complex root subspaces, are

$$L^{(i)}(a) = \left(L_c^{(i)}(a) + L_c^{(i)}(\overline{a})\right) \bigcap L;$$

components of $L^{(i)}$ corresponding to the real eigenvalues of $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$, are

$$L_r^{(i)} = \sum\nolimits^{\oplus} \left\{ L^{(i)}(a) \mid a \in \mathrm{Sp}_r^{(i)} \right\}.$$

The subalgebras $L^{(1)}$ and $L^{(2)}$ are ideals of L, so they are $(\operatorname{ad} B)$-invariant, and the restrictions $\operatorname{ad} B|_{L^{(1)}}$ and $\operatorname{ad} B|_{L^{(2)}}$ are well defined.

In the following lemma, we collect several simple statements about decomposition of the subalgebras $L^{(1)}$ and $L^{(2)}$ into sums of invariant subspaces of the adjoint operator $\operatorname{ad} B$.

Lemma 14.1.

(1) $L^{(i)} = \sum^{\oplus} \left\{ L^{(i)}(a) \mid a \in \mathrm{Sp}^{(i)},\ \operatorname{Im} a \geq 0 \right\}$, $i = 1, 2$;

(2) $\mathrm{Sp}^{(2)} \subset \mathrm{Sp}^{(1)}$, $\mathrm{Sp}_r^{(2)} \subset \mathrm{Sp}_r^{(1)}$;

(3) $L^{(2)}(a) \subset L^{(1)}(a)$ *for any* $a \in \mathrm{Sp}^{(2)}$;

(4) $L_r^{(2)} \subset L_r^{(1)}$;

(5) $\mathrm{Sp}^{(2)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$.

Proof. It is obtained by the standard linear-algebraic arguments. In addition, Jacobi's identity is applied in item (5).

Consider the quotient operator

$$\widetilde{\operatorname{ad}} B \ : \ L^{(1)}/L^{(2)} \to L^{(1)}/L^{(2)}$$

defined as follows:

$$\left(\widetilde{\operatorname{ad}} B\right)\left(X + L^{(2)}\right) = (\operatorname{ad} B)X + L^{(2)} \quad \forall X \in L^{(1)}.$$

Analogously, for $a \in \operatorname{Sp}^{(1)}$, we define the quotient operator in the quotient root space:

$$\widetilde{\operatorname{ad}} B(a) \ : \ L^{(1)}(a)/L^{(2)}(a) \to L^{(1)}(a)/L^{(2)}(a),$$
$$\left(\widetilde{\operatorname{ad}} B(a)\right)\left(X + L^{(2)}(a)\right) = (\operatorname{ad} B)X + L^{(2)}(a) \quad \forall X \in L^{(1)}(a),$$

and its complexification:

$$\widetilde{\operatorname{ad}_c} B(a) \ : \ L_c^{(1)}(a)/L_c^{(2)}(a) \to L_c^{(1)}(a)/L_c^{(2)}(a),$$
$$\left(\widetilde{\operatorname{ad}_c} B(a)\right)\left(X + L_c^{(2)}(a)\right) = (\operatorname{ad}_c B)\, X + L_c^{(2)}(a) \quad \forall X \in L_c^{(1)}(a).$$

Definition 14.1. Let $a \in \operatorname{Sp}^{(1)}$. Denote by $j(a)$ the geometric multiplicity of the eigenvalue a of the operator $\widetilde{\operatorname{ad}_c} B(a)$ in the vector space $L_c^{(1)}(a)/L_c^{(2)}(a)$.

Remarks.

(a) For $a \in \operatorname{Sp}^{(1)}$, the number $j(a)$ is equal to the number of Jordan blocks of the operator $\widetilde{\operatorname{ad}} B(a)$ in the space $L^{(1)}(a)/L^{(2)}(a)$.

(b) If an eigenvalue $a \in \operatorname{Sp}^{(1)}$ is simple, then $j(a) = 0$ for $a \in \operatorname{Sp}^{(2)}$ and $j(a) = 1$ for $a \in \operatorname{Sp}^{(1)} \setminus \operatorname{Sp}^{(2)}$.

Assume that $L = L^{(1)} \oplus \mathbb{R}B$ for some B in L (this assumption will be justified by Theorem 14.1 below). Then, by Lemma 14.1,

$$L = \mathbb{R}B \oplus L^{(1)} = \mathbb{R}B \oplus \sum\nolimits^{\oplus} \left\{ L^{(1)}(a) \mid a \in \operatorname{Sp}^{(1)},\ \operatorname{Im} a \geq 0 \right\}, \tag{14.2}$$

that is why any element $X \in L$ can uniquely be decomposed as follows:

$$X = X_B + \sum \left\{ X(a) \mid a \in \operatorname{Sp}^{(1)},\ \operatorname{Im} a \geq 0 \right\}, \quad X_B \in \mathbb{R}B,\ X(a) \in L^{(1)}(a).$$

We will consider such decomposition for the uncontrolled vector field A of the system Γ:

$$A = A_B + \sum \left\{ A(a) \mid a \in \operatorname{Sp}^{(1)},\ \operatorname{Im} a \geq 0 \right\}.$$

Denote by $\widetilde{A}(a)$ the canonical projection of the vector $A(a) \in L^{(1)}(a)$ onto the quotient space $L^{(1)}(a)/L^{(2)}(a)$.

Definition 14.2. Let $L = L^{(1)} \oplus \mathbb{R}B$, $a \in \mathrm{Sp}^{(1)}$, and let $j(a) = 1$. We say that a vector $A \in L$ *has the zero a-top* if

$$\widetilde{A}(a) \in \left(\widetilde{\mathrm{ad}}\, B(a) - a\,\mathrm{Id}\right)\left(L^{(1)}(a)/L^{(2)}(a)\right).$$

In the opposite case, we say that A *has a nonzero a-top.* We use the corresponding notations: $\mathrm{top}(A, a) = 0$ or $\mathrm{top}(A, a) \neq 0$.

Remark. Geometrically, if a vector A has a nonzero a-top, then the vector $\widetilde{A}(a)$ has a nonzero component corresponding to the highest adjoint vector in the (single) Jordan chain of the operator $\widetilde{\mathrm{ad}}\, B(a)$. Due to nonuniqueness of the Jordan base, this component is nonuniquely determined, but its property to be zero is basis-independent.

Definition 14.3. A pair of complex numbers (α, β), $\mathrm{Re}\,\alpha \leq \mathrm{Re}\,\beta$, is called an *$N$-pair of eigenvalues of the operator* $\mathrm{ad}\, B$ if the following conditions hold:

(1) $\alpha, \beta \in \mathrm{Sp}^{(1)}$,

(2) $L^{(2)}(\alpha) \not\subset \sum \left\{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)}, \quad \mathrm{Re}\, a,\ \mathrm{Re}\, b \notin [\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta] \right\}$,

(3) $L^{(2)}(\beta) \not\subset \sum \left\{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)}, \quad \mathrm{Re}\, a,\ \mathrm{Re}\, b \notin [\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta] \right\}$.

Remark. In other words, to generate both root spaces $L^{(2)}(\alpha)$ and $L^{(2)}(\beta)$ for an N-pair (α, β), we need at least one root space $L^{(1)}(\gamma)$ with $\mathrm{Re}\,\gamma \in [\alpha, \beta]$. In Theorem 14.2 below, N-pairs are the strongest obstruction to controllability under the necessary conditions of Theorem 14.1. In some generic cases, the property of absence of real N-pairs can be verified by using Lemma 14.5.

14.2. Necessary controllability conditions.

14.2.1. Formulation of results. It turns out that the controllability on simply-connected Lie groups G with $G \neq G^{(1)}$ is a very strong property: it imposes essential restrictions both on the group G and on the system Γ.

Theorem 14.1. *Let a connected Lie group G be simply-connected, and let its Lie algebra L satisfy the condition $L \neq L^{(1)}$. If a right-invariant system $\Gamma \subset L$ is controllable, then*

(1) $\dim L^{(1)} = \dim L - 1$;

(2) $B \notin L^{(1)}$;

(3) $L^{(2)}_r = L^{(1)}_r$;

(4) $\mathrm{Sp}^{(2)}_r = \mathrm{Sp}^{(1)}_r$;

(5) $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$;

(6) $j(a) \leq 1$ *for all* $a \in \mathrm{Sp}^{(1)}$;

(7) $\mathrm{top}(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)}$ *for which* $j(a) = 1$.

The notations $j(a)$ and $\mathrm{top}(A, a)$ used in Theorem 14.1 are introduced in Definitions 14.1 and 14.2.

Remarks.

(a) The first condition is a characterization of the state space G but not of the system Γ. It means that no single-input system $\Gamma = \{ A + uB \}$ can be controllable on a simply-connected Lie group G with $\dim G^{(1)} < \dim G - 1$. That is, to control on such a group, one has to increase the number of inputs. This agrees with the general lower estimate $m > \dim G - \dim G^{(1)}$ for the number of the controlled vector fields $B_1, \ldots, B_m$ that are necessary for controllability of the multi-input system $\Gamma = \{ A + \sum_{i=1}^{m} u_i B_i \}$ on a simply-connected group G, see Theorem 9.3.

(b) Conditions (3)–(7) are nontrivial only for Lie algebras L with $L^{(2)} \neq L^{(1)}$ (in particular, for solvable noncommutative L). If $L^{(2)} = L^{(1)}$, then these conditions are obviously satisfied.

(c) The third condition means that $j(a) = 0$ for all $a \in \mathrm{Sp}_r^{(1)}$; that is why condition (6) is nontrivial only for $a \in \mathrm{Sp}_c^{(1)}$.

(d) By the same reason, the inclusion $a \in \mathrm{Sp}^{(1)}$ in condition (7) can be replaced by $a \in \mathrm{Sp}_c^{(1)}$. Note that if $j(a) = 0$, then, by the formal Definition 14.2, the vector A has the zero a-top.

(e) The fourth and fifth conditions are implied by the third one but are easier to verify. The simple (and strong) "arithmetic" necessary controllability condition (5) can be verified by considering the spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$.

(f) For solvable Lie algebras L, under conditions (1) and (2), the spectrum $\mathrm{Sp}^{(1)} = \mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}})$ is the same for all $B \notin L^{(1)}$ modulo homothety. Then conditions (4) and (5) depend on L but not on B.

(g) For the case of a simple spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$, the necessary controllability conditions take respectively the more simple form.

Corollary 14.1. *Let a Lie group G be simply-connected, and let its Lie algebra L satisfy the condition $L \neq L^{(1)}$. Assume that the spectrum $\mathrm{Sp}^{(1)}$ is simple. If a right-invariant system $\Gamma \subset L$ is controllable, then*

(1) $\dim L^{(1)} = \dim L - 1$,

(2) $B \notin L^{(1)}$,

(3) $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$,

(4) $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$,

(5) $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)}$.

14.2.2. Outline of the proof of Theorem 14.1. The main tools for obtaining the necessary controllability conditions given in Theorem 14.1 are the rank controllability condition (Theorem 2.3) and the hypersurface principle (Theorem 12.2).

First, the following auxiliary propositions are proved.

Lemma 14.2. *Let L be a Lie algebra such that $L \neq L^{(1)}$, and let $B \in L$. Assume that*

(1) $\dim L^{(1)} < \dim L - 1$ *or*

(2) $B \in L^{(1)}$ *or*

(3) $L^{(1)} \oplus \mathbb{R}B = L$ *and* $L_r^{(2)} \neq L_r^{(1)}$.

Then there exists a codimension one subalgebra of L containing B.

Lemma 14.3. *Let L be a Lie algebra, and let $A, B \in L$. Let $L = \mathbb{R}B \oplus L^{(1)}$. Assume that there exists an eigenvalue $a \in \mathrm{Sp}^{(1)}$ such that*

(1) $j(a) > 1$ *or*

(2) $j(a) = 1$ *and* $\mathrm{top}(A, a) = 0$.

Then $\mathrm{Lie}(A, B) \neq L$.

Now Theorem 14.1 follows. If one of its conditions (1)–(5) is violated, then, by Lemma 14.2 and Theorem 12.2, the reachable set $\mathbb{A}$ is contained in the closed semigroup of the Lie group G bounded by a codimension one subgroup of G. If one of the conditions (6) and (7) does not hold, then, by Lemma 14.3 and Theorem 2.3, the set $\mathbb{A}$ lies in a proper connected subgroup of G with the Lie algebra $\mathrm{Lie}(\Gamma)$.

14.3. Sufficient controllability conditions.

14.3.1. Formulation of results. Under the necessary assumptions of Theorem 14.1, there exist many sufficient controllability conditions. Notice that the assumption on the simple connectedness can now be removed. So, the sufficient conditions below are completely Lie-algebraic, i.e., local; this is in contrast to the global assumption (the finiteness of center of G) essential for sufficient controllability conditions for semisimple Lie groups G (see Sec. 8).

Theorem 14.2. *Let $\Gamma \subset L$ be a right-invariant system on a connected Lie group G. Assume that the following conditions hold*:

(1) $\dim L^{(1)} = \dim L - 1$;

(2) $B \notin L^{(1)}$;

(3) $L_r^{(2)} = L_r^{(1)}$;

(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(5) $\mathrm{top}(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(6) *the operator* $\mathrm{ad}\, B|_{L^{(1)}}$ *has no N-pairs of real eigenvalues.*

Then the system Γ is controllable on the Lie group G.

The notation $\mathrm{top}(A, a)$ and the notion of N-pair used in Theorem 14.2 are introduced in Definitions 14.2 and 14.3.

Remarks.

(a) Conditions (1)–(3) are necessary for controllability in the case of a simply connected $G \neq G^{(1)}$; see Theorem 14.1.

(b) Conditions (4) and (5) are close to the necessary conditions (6) and (7) of Theorem 14.1, respectively. Notice that the fourth condition means that all complex eigenvalues of $\mathrm{ad}\, B|_{L^{(1)}}$ are geometrically simple.

(c) Conditions (2) and (5) are open, i.e., they are preserved under small perturbations of A and B.

(d) The most restrictive of the conditions (1)–(6) is the last one. It can be shown that the smallest dimension of $L^{(1)}$ in which this condition is satisfied and preserved under small perturbations of spectrum of $\mathrm{ad}\, B|_{L^{(1)}}$ for solvable L is 6. This can be used to obtain a classification of controllable systems Γ on lower-dimensional simply connected solvable Lie groups G; see Sec. 16.

(e) The technically complicated condition (6) can be replaced with a more simple and more restrictive one, and sufficient conditions can be given as in Corollary 14.2 below.

(f) Under the additional assumption of simplicity of the spectrum $\mathrm{Sp}^{(1)}$, the sufficient controllability conditions take the even more simple form presented in Corollary 14.3 below.

Corollary 14.2. *Assume that the following conditions hold for a system* $\Gamma \subset L$ *on a Lie group* G:

(1) $\dim L^{(1)} = \dim L - 1$;

(2) $B \notin L^{(1)}$;

(3) $L_r^{(2)} = L_r^{(1)}$;

(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(5) $\mathrm{top}(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(6) $\mathrm{Sp}_r^{(1)} = \varnothing$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z > 0\}$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z < 0\}$.

Then the system Γ is controllable on G.

Corollary 14.3. *Assume that the following conditions hold for a system* $\Gamma \subset L$ *on a Lie group* G:

(1) $\dim L^{(1)} = \dim L - 1$;

(2) $B \notin L^{(1)}$;

(3) *the spectrum* $\mathrm{Sp}^{(1)}$ *is simple*;

(4) $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$;

(5) $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(6) $\mathrm{Sp}_r^{(1)} = \varnothing$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z > 0\}$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z < 0\}$.

Then the system Γ is controllable on G.

14.3.2. Outline of the proof of Theorem 14.2. This theorem is obtained via the Lie saturation technique; see Sec. 4: a sequence of increasing lower bounds of the tangent cone $\mathrm{LS}(\Gamma)$ to the closure of the attainable set $\mathbb{A}$ at the identity e is shown to stabilize at the whole Lie algebra L.

The crucial role in the proof is played by the following proposition.

Lemma 14.4. *Let* $C \in \mathrm{LS}(\Gamma) \cap L^{(1)}$. *Assume that for any* $a \in \mathrm{Sp}_c^{(1)}$, *the following conditions hold*:

(1) $\dim L_c(a) = 1$ *and*

(2) $\mathrm{top}(C, a) \neq 0$ *or* $L^{(1)}(a) \subset \mathrm{LS}(\Gamma)$.

Assume additionally that for the number

$$r = \max\{\operatorname{Re} a \mid a \in \mathrm{Sp}^{(1)},\ C(a) \neq 0\}$$

$$(\textit{or } r = \min\{\operatorname{Re} a \mid a \in \mathrm{Sp}^{(1)},\ C(a) \neq 0\}),$$

we have $r \notin \mathrm{Sp}^{(1)}$ *or* $C(r) = 0$. *Then*

$$\mathrm{LS}(\Gamma) \supset \sum \left\{ L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)},\ \operatorname{Re} a = r,\ a \neq r \right\}.$$

Now the idea of the proof of Theorem 14.2 can be outlined as follows. In view of (14.2), the Lie algebra L splits into the direct sum of the line $\mathbb{R}B$ and the root spaces $L^{(1)}(a)$, $a \in \mathrm{Sp}^{(1)}$. We show that the Lie saturate $\mathrm{LS}(\Gamma)$ coincides with L. First of all, it easy to see that $\mathbb{R}B \subset \mathrm{LS}(\Gamma)$. Then we prove on the contrary that $L^{(1)}(a) \subset \mathrm{LS}(\Gamma)$ for all $a \in \mathrm{Sp}^{(1)}$. Indeed, assume that there exist numbers

$$n = \min \left\{ \operatorname{Re} a \mid a \in \mathrm{Sp}^{(1)},\ L^{(1)}(a) \not\subset \mathrm{LS}(\Gamma) \right\},$$

$$m = \max \left\{ \operatorname{Re} a \mid a \in \mathrm{Sp}^{(1)},\ L^{(1)}(a) \not\subset \mathrm{LS}(\Gamma) \right\}$$

and consider the closed interval $[n, m] \subset \mathbb{R}$. Then Lemma 14.4 implies that n, m is an N-pair of real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$; this contradicts hypothesis (6) of Theorem 14.2. The theorem is proved.

Corollaries 14.2 and 14.3 follow from Theorem 14.2 and the following proposition, which gives simple conditions that guarantee the nonexistence of real N-pairs of eigenvalues.

Lemma 14.5. *Assume that* $B \notin L^{(1)}$ *and* $L_r^{(1)} = L_r^{(2)}$. *Then any one of the following conditions is sufficient for the operator* $\operatorname{ad} B|_{L^{(1)}}$ *not to have real* N*-pairs of eigenvalues*:

(1) $\mathrm{Sp}_r^{(1)} = \varnothing$ *or*

(2) $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$ *or*

(3) $\mathrm{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

The controllability conditions of Theorems 14.1 and 14.2 for Lie groups $G \neq G^{(1)}$ yield a complete description of controllability for several particular classes of Lie groups: meta-Abelian ones, some subgroups of the group of affine transformations of $\mathbb{R}^n$, and lower-dimensional simply-connected solvable Lie groups. These results are presented in Secs. 15 and 16.

14.4. Remarks. The results of this section are due to Sachkov [120].

The results of Hofmann [64] on compact elements in solvable Lie algebras might be helpful in order to understand the controllability on solvable Lie groups without the assumption on the simple connectedness that is essential for necessary controllability conditions in this section.

15. Meta-Abelian Lie Groups

Lie algebras L having derived series of length 2:

$$L \supset L^{(1)} \supset L^{(2)} = \{0\},$$

are called *meta-Abelian.* A Lie group with a meta-Abelian Lie algebra is also called *meta-Abelian.*

A meta-Abelian Lie algebra is obviously solvable. Thus results of the previous section yield controllability conditions for meta-Abelian Lie groups.

Theorem 15.1. *Let G be a meta-Abelian connected Lie group. Then the following conditions are sufficient for controllability of a system $\Gamma = A + \mathbb{R}B \subset L$ on G:*

(1) $\dim L^{(1)} = \dim L - 1$;

(2) $B \notin L^{(1)}$;

(3) $\mathrm{Sp}_r^{(1)} = \varnothing$;

(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$;

(5) $\mathrm{top}(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$.

If the group G is simply connected, then conditions (1)–(5) are also necessary for controllability of the system Γ on G.

The notation $\mathrm{top}(A, a)$ was introduced in Definition 14.2.

Proof. The sufficiency follows from Corollary 14.2.

In order to prove the necessity for the simply-connected G, assume that Γ is controllable. Then (1) and (2) follow from items (1) and (2) of Theorem 14.1.

Condition (3) follows from item (3) of Theorem 14.1 and from the meta-Abelian property of G:

$$L_r^{(1)} = L_r^{(2)} \subset L^{(2)} = \{0\}.$$

Condition (4). For any $a \in \mathrm{Sp}_c^{(1)}$, we have $L^{(2)}(a) = \{0\}$; that is why $j(a)$ is equal to geometric multiplicity of the eigenvalue a of the operator $\mathrm{ad}\, B|_{L^{(1)}(a)}$, i.e., to $\dim L_c(a)$. By item (6) of Theorem 14.1, we have $j(a) = 1$; that is why $\dim L_c(a) = 1$.

Condition (5). For any $a \in \mathrm{Sp}_c^{(1)}$, we have $j(a) = 1$; then, by item (7) of Theorem 14.1, $\mathrm{top}(A, a) \neq 0$.

Example 15.1. Let V be a finite-dimensional real vector space, and let $l \subset \mathfrak{gl}(V)$ be a linear Lie algebra. Consider their semidirect sum $L = V \rightthreetimes l$. It is a subalgebra of the Lie algebra of the group of affine transformations of the space V since $L \subset V \rightthreetimes \mathfrak{gl}(V)$. If l is Abelian, then L is meta-Abelian:

$$L^{(1)} = lV \rightthreetimes \{0\}, \quad L^{(2)} = \{0\}.$$

In the next subsection, we study in detail a particular case where l is one-dimensional.

15.1. Semidirect products. Let V be a real finite-dimensional vector space, $\dim V = n$, and let M be a nonzero linear operator in V. Consider the meta-Abelian Lie algebra $L(M)$, which is the semidirect sum of the Abelian Lie algebra V with the one-dimensional Lie algebra $\mathbb{R}M$. This Lie algebra can be represented by $(n+1) \times (n+1)$ matrices:

$$L(M) = \left\{ \begin{pmatrix} Mt & b \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{R},\ b \in \mathbb{R}^n \right\} \subset \mathfrak{gl}(n+1; \mathbb{R}). \tag{15.1}$$

Denote by $G(M)$ the connected Lie subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ corresponding to $L(M)$. It is the semidirect product of the vector Lie group $\mathbb{R}^n$ with the one-dimensional Lie group $G_1 = \{ \exp(Mt) \mid t \in \mathbb{R} \}$. Elements of the group $G(M)$ are the matrices

$$\begin{pmatrix} \exp(Mt) & p \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\ p \in \mathbb{R}^n;$$

thus, $G(M)$ can be viewed as a subgroup of the group $\mathrm{Aff}(n;\mathbb{R})$ of affine transformations of $\mathbb{R}^n$ generated by the one-parameter group of automorphisms G_1 and all translations $p \in \mathbb{R}^n$. The group $G(M)$ is not simply-connected iff the one-parameter subgroup G_1 is periodic; this obviously occurs iff

$$\begin{cases} \text{the matrix } M \text{ is semisimple}, \\ \mathrm{Sp}(M) = ir \cdot (k_1, \dots, k_n) \quad \text{for some } r \in \mathbb{R},\ (k_1, \dots, k_n) \in \mathbb{Z}^n. \end{cases} \tag{15.2}$$

Remark. If conditions (15.2) hold, then, by Theorem 7.1, on semidirect products of vector spaces with compact Lie groups, a system $\Gamma \subset L(M)$ is controllable on $G(M)$ if and only if it has a full rank: $\mathrm{Lie}(\Gamma) = L(M)$.

On the other hand, the controllability test for simply-connected meta-Abelian Lie groups (Theorem 15.1) implies the following controllability conditions for the universal covering $\tilde{G}(M)$ and for the group $G(M)$ itself.

Theorem 15.2. *Let M be a nonzero $n \times n$ matrix, $G = \tilde{G}(M)$, and let $L = L(M)$. A system $\Gamma = A + \mathbb{R}B \subset L$ is controllable on G if and only if the following conditions hold*:

(1) *the matrix M has a purely complex spectrum*;

(2) $B \notin L^{(1)}$;

(3) $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, \dots, (\mathrm{ad}\, B)^{n-1}A) = L$.

For the group $G(M)$, conditions (1)–(3) *are sufficient for controllability. If conditions* (15.2) *are violated, then* (1)–(3) *are equivalent to the controllability on $G(M)$.*

Example 15.2. Let $G = \mathrm{E}(2;\mathbb{R})$ be the Euclidean group of motions of the plane $\mathbb{R}^2$. Its Lie algebra $L = \mathfrak{e}(2;\mathbb{R})$ is spanned by the matrices $A_1 = E_{13}$, $A_2 = E_{23}$, and $A_3 = E_{21} - E_{12}$ and has form (15.1):

$$L = L(M), \quad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is solvable (in fact, meta-Abelian):

$$L^{(1)} = \mathrm{span}(A_1, A_2) \supset L^{(2)} = \{0\},$$

but not completely solvable:

$$\mathrm{Sp}(\mathrm{ad}\, A_3) = \{\pm i,\, 0\}.$$

The Lie group $\mathrm{E}(2;\mathbb{R}) = G(M)$ is connected but not simply-connected; compare with (15.2).

Consider the system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ on $\tilde{\mathrm{E}}(2;\mathbb{R})$, the simply-connected covering of $\mathrm{E}(2;\mathbb{R})$. A complete characterization of controllability of Γ on $\tilde{\mathrm{E}}(2;\mathbb{R})$ is derived from Theorem 15.2.

Theorem 15.3. *A system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ is controllable on $\tilde{\mathrm{E}}(2;\mathbb{R})$ if and only if the vectors A and B are linearly independent and $B \notin \mathrm{span}(A_1, A_2)$.*

Compare the controllability conditions for $\tilde{\mathrm{E}}(2;\mathbb{R})$ with the following conditions for $\mathrm{E}(2;\mathbb{R})$ derived from Theorem 7.1.

Theorem 15.4. *A system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ is controllable on $\mathrm{E}(2;\mathbb{R})$ if and only if the vectors A and B are linearly independent and $\{A, B\} \not\subset \mathrm{span}(A_1, A_2)$.*

15.2. Affine systems. Given any matrix $A \in \mathrm{M}(n;\mathbb{R})$ and any vector $b \in \mathbb{R}^n$, consider the affine system

$$\dot{x} = uAx + b, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}. \tag{Σ}$$

According to Sec. 3.3, such system is subordinated to the linear action of the group $G(A) \subset \mathrm{Aff}(n;\mathbb{R})$ described in the previous subsection.

This observation in combination with the controllability results for right-invariant systems on Lie groups of the form $G(A)$ lead us to complete controllability conditions for affine systems Σ.

Theorem 15.5. *The system Σ is globally controllable on $\mathbb{R}^n$ if and only if the following conditions hold:*

(1) *the matrix A has a purely complex spectrum and*

(2) $\mathrm{span}(b, Ab, \dots, A^{n-1}b) = \mathbb{R}^n$.

Proof. *Sufficiency.* Consider the right-invariant system $\Gamma = \overline{A} + \mathbb{R}\overline{B} \subset L(A)$ on the Lie group $G(A)$, where the matrices $\overline{A}, \overline{B} \in L(A)$ are given by

$$\overline{A} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

The affine system Σ is induced by the right-invariant system Γ. On the other hand, the group of affine transformations $G(A) \subset \mathrm{Aff}(n;\mathbb{R})$ acts transitively on $\mathbb{R}^n$, since it contains all translations. By Corollary 3.3, if the right-invariant system Γ is controllable on $G(A)$, then the affine system Σ is controllable on $\mathbb{R}^n$. By Theorem 15.2, the system Γ is controllable on $G(A)$; thus, the sufficiency follows.

Necessity. If one of the conditions (1) and (2) of Theorem 15.5 is violated, then the system Σ has codimension 1 or 2 invariant subspaces in $\mathbb{R}^n$.

15.3. Remarks. The results of this section were obtained by Sachkov [120].

16. Small-Dimensional Simply Connected Solvable Lie Groups

Given a Lie algebra L, there is the "largest" connected Lie group G having Lie algebra L, the simply-connected one. All other connected Lie groups with Lie algebra L are "smaller" than G in the sense that they are quotients G/C, where C is a discrete subgroup of center of G. A right-invariant system $\Gamma \subset L$ can thus be considered on any of these groups, and the simply-connected group G is the most difficult to control among them. Hence, given a right-invariant system Γ on a Lie group (or a homogeneous space of a Lie group) H, it is natural first to study its controllability on the simply-connected covering $\widetilde{H}$ of H. If Γ is controllable on $\widetilde{H}$, then it is obviously controllable on H (and on all its homogeneous spaces); in the opposite case, one should use particular geometric properties of H (e.g., the existence of periodic one-parameter subgroups) to verify the controllability of Γ on H. It is obvious and remarkable that controllability conditions on a simply-connected Lie group G should have a completely Lie-algebraic form: they are completely determined by the Lie algebra L and its subset Γ (see, e.g., Theorems 9.4, 13.1, 14.1, 15.1, and 15.2).

This motivates the following definition.

Definition 16.1. A system $\Gamma \subset L$ is called *controllable* if it is controllable on a (unique) connected simply-connected Lie group with Lie algebra L.

The next definition makes sense at least for solvable Lie algebras in lower dimensions.

Definition 16.2. A Lie algebra L is called *controllable* if there exist $A, B \in L$ such that the system $\Gamma = A + \mathbb{R}B$ is controllable.

Indeed, it turns out that controllability conditions on solvable Lie groups (Secs. 11 and 14) imply that for solvable lower-dimensional Lie algebras L,

(1) the existence of a controllable single-input system $\Gamma \subset L$, i.e., controllability of L, is a strong restriction on L;

(2) if L is controllable, then almost all pairs $(A, B) \in L \times L$ give rise to controllable systems $\Gamma = A + \mathbb{R}B$;

(3) the controllability of a system $\Gamma \subset L$ depends primarily on L but not on Γ.

Moreover, these results yield a complete description of controllability in lower-dimensional solvable Lie algebras presented in the following subsections.

Up to dimension 6, we describe all solvable Lie algebras L that are controllable, and give controllability tests for single-input systems $\Gamma = A + \mathbb{R}B \subset L$ (the only gap in this picture is the class $L_{6,IV}$ of six-dimensional Lie algebras not completely studied).

The general "bird's-eye view" of controllable small-dimensional solvable Lie algebras is as follows:

$\dim L = 1$ the (unique) Lie algebra is controllable;

$\dim L = 2$ the two Lie algebras are noncontrollable;

$\dim L = 3$ there is one family of controllable Lie algebras $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\dim L = 4$ there is one family of controllable Lie algebras $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\dim L = 5$ there are two families of controllable Lie algebras:

1. $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\dim L = 6$ there are five families of controllable Lie algebras:

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$,
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,

and one exceptional class $L_{6,IV}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, containing both controllable and noncontrollable Lie algebras.

All controllable Lie algebras L are presented by a scheme in the complex plane $\mathbb{C}$ containing eigenvalues of the adjoint operator $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, and arrows between these eigenvalues describing Lie brackets between eigenvectors of the operator $\operatorname{ad} B|_{L^{(1)}}$ (these schemes are given at the very end of this section). Notice that for solvable Lie algebras L with codimension one subalgebras $L^{(1)}$ (and only such solvable Lie algebras may be controllable, see condition (1) of Theorem 14.1), spectra of all adjoint operators $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, are homothetic with respect to the origin $0 \in \mathbb{C}$, and the homothety equivalence class of spectra of $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, is determined not by $B \in L \setminus L^{(1)}$ but by L itself (in fact, by the isomorphism class of L).

Now we present the classification of controllability in lower-dimensional solvable Lie algebras. These results are obtained by virtue of controllability conditions of Secs. 11, 12, and 14. The proofs are outlined up to the first nontrivial dimension 3: for dimensions 4–6 the idea of proofs is analogous to the 3-dimensional case but the argument is much longer.

16.1. One-dimensional Lie groups. A unique one-dimensional Lie algebra is Abelian and isomorphic to $\mathbb{R}$.

Theorem 16.1. *The one-dimensional Lie algebra $\mathbb{R}$ is controllable.*
A system $\Gamma = A + \mathbb{R}B \subset \mathbb{R}$ is controllable if and only if $B \neq 0$.

Proof. Apply Corollary 9.1.

16.2. Two-dimensional Lie groups. There are two nonisomorphic two-dimensional Lie algebras: the Abelian $\mathbb{R}^2$ and the solvable non-Abelian $S_2 = \operatorname{span}(x, y)$, $[x, y] = y$.

Theorem 16.2. *Both two-dimensional Lie algebras $\mathbb{R}^2$ and S_2 are not controllable.*

Proof. Both $\mathbb{R}^2$ and S_2 are completely solvable; thus, Theorem 13.1 can be applied.

16.3. Three-dimensional Lie groups.

Construction 16.1. The Lie algebra $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (Fig. 2).

$$L_3(\lambda) = \operatorname{span}(x, y, z),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z)} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The Lie algebra $L_3(\lambda)$ is schematically represented in Fig. 2 by the eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$ and realifications of the eigenvectors $y, z \in L_3(\lambda)$ of the adjoint operator $\operatorname{ad} x|_{\operatorname{span}(y,z)}$.

Theorem 16.3. *Let $L = L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:

(1) *$B \notin L^{(1)}$,*

(2) *the vectors A and B are linearly independent.*

Proof. *Sufficiency.* We show that all the hypotheses of Corollary 14.3 hold.

Conditions (1) and (2) are obviously satisfied.

Condition (3). Consider the decomposition $B = B_x x + B_y y + B_z z$. We have

$$\mathrm{Sp}^{(1)} = \mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}}) = B_x \cdot \mathrm{Sp}(\mathrm{ad}\, x|_{L^{(1)}}) = B_x \cdot \{\lambda, \bar{\lambda}\}.$$

$B \notin L^{(1)}$ is equivalent to $B_x \neq 0$; thus, the spectrum $\mathrm{Sp}^{(1)}$ is simple.

Condition (4): $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)} = \varnothing$.

Condition (5), $A(a) \neq 0$ for all $a \in \mathrm{Sp}_c^{(1)}$, means that the vector A has a nonzero projection onto $L^{(1)}$ along the line $\mathbb{R}B$, i.e., that A and B are linearly independent.

Condition (6): $\mathrm{Sp}_r^{(1)} = \varnothing$.

Now it follows from Corollary 14.3 that the system Γ is controllable.

The necessity follows from Corollary 14.1.

Theorem 16.4. *A three-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to* $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. *Sufficiency.* The set of systems Γ that satisfy conditions (1) and (2) of Theorem 16.3 is nonempty.

Necessity. Let $\Gamma = A + \mathbb{R}B \subset L$ be a controllable system. By Theorem 14.1, $\dim L^{(1)} = 2$ and $B \notin L^{(1)}$. The derived subalgebra $L^{(1)}$ is nilpotent and two-dimensional; thus, it is Abelian. Consequently, $\mathrm{Sp}_r^{(2)} \subset \mathrm{Sp}^{(2)} = \varnothing$. Thus,

$$\mathrm{Sp}^{(1)} = \mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}}) = \{\lambda, \bar{\lambda}\}, \quad \lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}.$$

Then there exist a basis y, z of the plane $L^{(1)}$ such that

$$[B, y] = \alpha y + \beta z, \quad [B, z] = -\beta y + \alpha z.$$

Taking into account that $L^{(1)}$ is Abelian, we obtain that $L \simeq L_3(\lambda)$.

16.4. Four-dimensional Lie groups.

Construction 16.2. The Lie algebra $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (Fig. 3).

$$L_4(\lambda) = \mathrm{span}(x, y, z, w),$$

$$\mathrm{ad}\, x|_{\mathrm{span}(y,z,w)} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y, z] = w.$$

The arrows in the schematic representation of the Lie algebra $L_4(\lambda)$ in Fig. 3 mean that the Lie bracket of the vectors y and z gives the vector w.

Theorem 16.5. *Let* $L = L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;

2. $A(\lambda) \neq 0$.

Theorem 16.6. *A four-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to* $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

16.5. Five-dimensional Lie groups.

Construction 16.3. The Lie algebra $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ (Fig. 4).

$$L_{5,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}, \quad \lambda = a + bi, \;\; \mu = c + di.$$

Construction 16.4. The Lie algebra $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (Fig. 5).

$$L_{5,II}(\lambda) = \operatorname{span}(x, y, z, u, v),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 1 & 0 & a & -b \\ 0 & 1 & b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The circles around the eigenvalues λ, $\bar{\lambda}$ in Fig. 5 mean that they have double algebraic multiplicity. (Notice that according to the previous matrix, their geometric multiplicity is simple.)

Theorem 16.7. *Let* $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;
2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

Theorem 16.8. *Let* $L = L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$,
2. $\operatorname{top}(A, \lambda) \neq 0$.

Remark. The notation $\operatorname{top}(A, \lambda) \neq 0$ in Theorem 16.8 (and in Theorem 16.14 below) means that the vector A has a nonzero component corresponding to the higher order root space of the operator $\operatorname{ad} B|_{L^{(1)}}$ corresponding to its eigenvalue λ.

Theorem 16.9. *A five-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, or $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

16.6. Six-dimensional Lie groups.

Construction 16.5. The Lie algebra $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ (Fig. 6).

$$L_{6,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & c & -d & 0 \\ 0 & 0 & d & c & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi, \ \mu = c + di,$$

$$[y, z] = w.$$

Construction 16.6. The Lie algebra $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $k \in \mathbb{R} \setminus \{0\}$ (Fig. 7).

$$L_{6,II}(\lambda, \mu, k) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & a & -d & 0 \\ 0 & 0 & d & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi, \ \mu = a + di,$$

$$[y, z] = w, \quad [u, v] = kw.$$

Construction 16.7. The Lie algebra $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$ (Fig. 8).

$$L_{6,III}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & 3a & -b & 0 \\ 0 & 0 & b & 3a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[w, y] = ku + lv, \quad [w, z] = -lu + kz.$$

Construction 16.8. The Lie algebra $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$ (Fig. 9).

$$L_{6,IV}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & -a & -b & 0 \\ 0 & 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y,v] = -[z,u] = kw, \quad [y,u] = [z,v] = lw.$$

Construction 16.9. The Lie algebra $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$ (Fig. 10).

$$L_{6,V}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 1 & 0 & a & -b & 0 \\ 0 & 1 & b & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y,z] = kw, \quad [y,u] = [z,v] = lw.$$

Construction 16.10. The class of Lie algebras $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$ (Fig. 11).

A Lie algebra L belongs to the class $L_{6,VI}(bi)$ if

$L = \operatorname{span}(x, y, z, u, v, w)$;

$L^{(1)} = \operatorname{span}(y, z, u, v, w)$;

$\operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = \{\pm bi, 0\}$;

both eigenvalues $\pm bi$ have double algebraic multiplicity;

$w \in L^{(2)}$.

The class $L_{6,VI}$ contains a lot of nonisomorphic Lie algebras in which multiplication can not be described in detail as in Lie algebras $L_{6,I}$–$L_{6,V}$.

Theorem 16.10. *Let $L = L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;
2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

Theorem 16.11. *Let $L = L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;

2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

Theorem 16.12. *Let* $L = L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;
2. $A(\lambda) \neq 0$.

Theorem 16.13. *Let* $L = L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$ *and* $A(-\lambda) \neq 0$.

Theorem 16.14. *Let* $L = L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable if and only if the following conditions hold*:

1. $B \notin L^{(1)}$;
2. $\operatorname{top}(A, \lambda) \neq 0$.

Remark. The class $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, contains both controllable and noncontrollable Lie algebras.

Theorem 16.15. *Let a six-dimensional solvable Lie algebra* L *do not belong to the class* $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$. *Then* L *is controllable if and only if it is isomorphic to one of the following Lie algebras*:

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$;
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda = \operatorname{Re}\mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$;
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$.

Controllable solvable Lie algebras up to dimension 6:

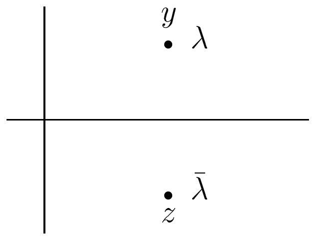

Fig. 2. $L_3(\lambda)$.

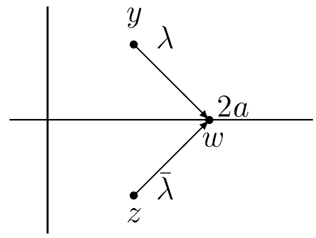

Fig. 3. $L_4(\lambda)$, $\operatorname{Re}\lambda = a$.

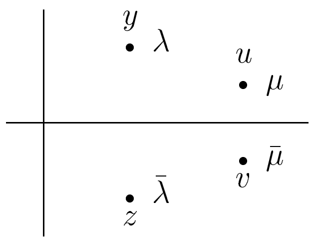

Fig. 4. $L_{5,I}(\lambda, \mu)$.

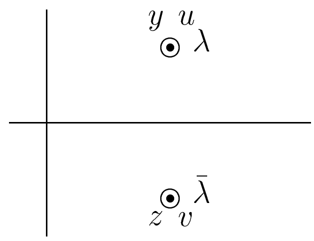

Fig. 5. $L_{5,II}(\lambda)$.

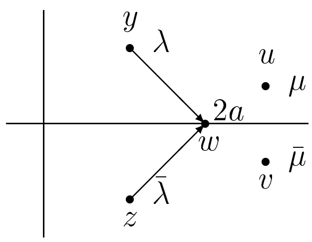

Fig. 6. $L_{6,I}(\lambda, \mu)$, $\operatorname{Re}\lambda = a$.

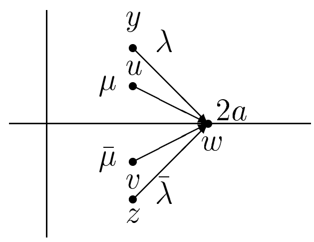

Fig. 7. $L_{6,II}(\lambda, \mu, k)$, $\operatorname{Re}\lambda = \operatorname{Re}\mu = a$.

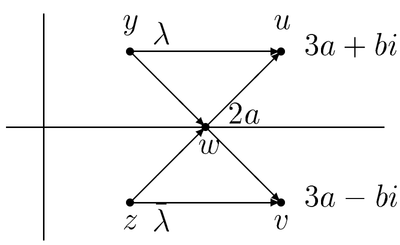

Fig. 8. $L_{6,III}(\lambda, k, l)$, $\lambda = a + bi$.

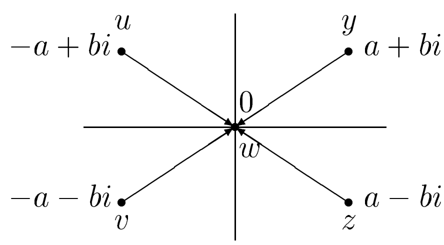

Fig. 9. $L_{6,IV}(\lambda, k, l)$, $\lambda = a + bi$.

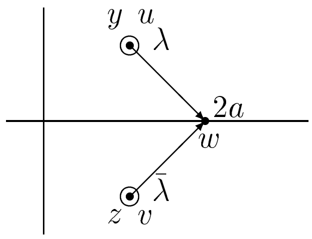

Fig. 10. $L_{6,V}(\lambda, \mu)$, $\operatorname{Re}\lambda = a$.

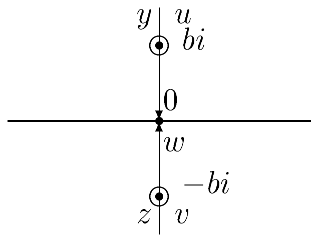

Fig. 11. $L_{6,VI}(bi)$.

16.7. Remarks. The classification of small-dimensional controllable solvable Lie algebras is due to Sachkov [122, 123].

A natural next step would be a complete and visual classification of controllable systems on general small-dimensional Lie algebras by synthesizing the "semi-simple" and "solvable" theory via Levi decomposition (Kupka [92]).

17. Final remarks

In this section, we collect some references related to the subject of this survey:

Surveys and reference works on controllability of right-invariant systems on Lie groups: Chong and Lawson [42], Kupka [92], Sachkov [122], Sallet [125, 126, 127].

Textbooks and surveys on geometric control theory: Agrachev, Vakhrameev, and Gamkrelidze [4], Andereev [6], Brockett [37], Casti [40], Gauthier [47], Jurdjevic [79], Sussmann [141], Vakhrameev [144], Vakhrameev and Sarychev [145].

Textbooks and surveys on Lie groups and Lie algebras: Bourbaki [34, 35], Varadarajan [146], Vinberg and Onishchik [147], Vinberg, Gorbatcevich, and Onishchik [148].

Textbooks on Lie semigroup theory: Hofmann and Lawson [66], Hilgert and Neeb [59], Hilgert, Hofmann and Lawson [58].

Controllability of nonlinear systems: Agrachev [3], Bacciotti and Stefani [23], Basto Gonçalves [18, 19, 20, 21], Bianchini and Stefani [24], Hermann [50], Hermes [51, 52, 53], Hermes and Kawski [54], Kawski [84], Krener [86], Levitt and Sussmann [95], Lobry [97, 98], Stefani [137], Sussmann [142], Sussmann and Jurdjevic [139], Tretyak [143].

Controllability of bilinear and affine systems: Adda [1], Adda and Sallet [2], Bacciotti [22], Bonnard [25, 26], Boothby [30], Brockett [38], Bruni, Di Pillo, and Koch [39], Koditschek and Narendra [85], Elliott and Tarn [45], Imbert, Clique, and Fossard [69], Joo and Tuan [72], Jurdjevic and Sallet [82], Kučera [89, 90, 91], Lepe [96], Lobry [99], Piechottka [111], Piechottka and Frank [112], Rink and Möhler [113], Sachkov [114, 115, 116, 117, 119].

Linear and bilinear systems on Lie groups: Ayala and Tirao [15], Ayala and Jiron [14], Ayala and Hacibekiroglu [16], Ayala, Rojo, and Soto [17], Markus [100].

Motion planning on Lie groups and their representation spaces: Krishnaprasad and Tsakiris [87, 88], Leonard [106], Leonard and Krishnaprasad [107, 108], Sarti, Walsh, and Sastry [131], Walsh, Montgomery, and Sastry [149], Zelikin [153, 154, 155, 156].

Control problems on Lie groups: Bonnard [27], Cheng, Dayawansa, and Martin [41], Enos [46], Hirschorn [61], Jurdjevic [75, 76, 77, 78], Lovric [109], Mittenhuber [101, 103], Monroy-Pérez [104], Sussmann [138], Yakovenko [150, 151], Yatcenko [152].

Acknowledgments

The author is grateful to professor A. A. Agrachev for the proposal of the research subject of right-invariant systems on Lie groups.

The author wishes to thank professors V. Ayala, B. Bonnard, F. Colonius, J. P. Gauthier, J. Hilgert, K. H. Hofmann, V. Jurdjevic, J .D. Lawson, D. Mittenhuber, L. San Martin, F. Silva Leite, and M. I. Zelikin for valuable discussions of the subject of this paper and for supplying their papers.

REFERENCES

1. Ph. Adda, "Contrôllabilité des systèmes bilinéaires generaux et homogenes dans $\mathbb{R}^2$," *Lect. Notes Contr. Inform. Sci.*, **111**, 205–214 (1988).

2. Ph. Adda and G. Sallet, "Determination algorothmique de la contrôllabilité pour des familles finies de champs de vecteurs lineaires sur $\mathbb{R}^2 \setminus \{0\}$," *R. A. I. R. O. APII*, **24**, 377–390 (1990).

3. A. A. Agrachev, "Local controllability and semigroups of diffeomorphisms," *Acta Appl. Math.*, **32**, 1–57 (1993).

4. A. A. Agrachev, S. V. Vakhrameev, and R. V. Gamkrelidze, "Differential-geometric and group-theoretic methods in optimal control theory," in: *Progress of Science and Technology, Series on Problems in Geometry* [in Russian], Vol. 14, All Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1983), pp. 3–56.

5. H. Albuquerque and F. Silva Leite, "On the generators of semisimple Lie algebras," *Linear Algebra Appl.*, **119**, 51–56 (1989).

6. Yu. N. Andreev, "Differential-geometric methods in control theory," *Automat. Telemekh.*, No. 10, 5–46 (1982).

7. R. El Assoudi, "Accessibilité par des champs de vecteurs invariants à droite sur un groupe de Lie," Thèse de doctorat de l'Université Joseph Fourier, Grenoble (1991).

8. R. El Assoudi and J. P. Gauthier, "Controllability of right invariant systems on real simple Lie groups of type F_4, G_2, C_n, and B_n," *Math. Control Signals Systems*, **1**, 293–301 (1988).

9. R. El Assoudi and J. P. Gauthier, "Controllability of right-invariant systems on semi-simple Lie groups," in: *New Trends in Nonlinear Control Theory*, Springer-Verlag **122** (1989); pp. 54–64.

10. R. El Assoudi, J. P. Gauthier, and I. Kupka, "On subsemigroups of semisimple Lie groups," *Ann. Inst. Henri Poincaré*, **13**, No. 1, 117–133 (1996).

11. L. Auslander, L. Green, and F. Hahn, "Flows on homogeneous spaces," *Ann. Math. Studies*, No. 53, Princeton Univ. Press, Princeton, New Jersey, (1963).

12. V. Ayala Bravo, "Controllability of nilpotent systems," in: *Geometry in nonlinear control and differential inclusions*, Banach Center Publications, Warszawa, **32** (1995), pp. 35–46.

13. V. Ayala Bravo and L. Vergara, "Co-adjoint representation and controllability," *Proyecciones* **11**, 37–48 (1992).

14. V. Ayala Bravo and I. Jiron, "Observabilidad del producto directo de sistemas bilineales," *Revista Cubo*, **9**, 35–46 (1993).

15. V. Ayala Bravo and J. Tirao, "Controllability of linear vector fields on Lie groups," Int. Centre Theor. Physics, *Preprint* IC/94/310, Trieste, Italy, (1994).

16. V. Ayala Bravo and A. Hacibekiroglu, "Observability of linear systems on Lie groups," Int. Centre Theor. Physics, *Preprint* IC/95/2, Trieste, Italy, (1995).

17. V. Ayala Bravo, O. Rojo, and R. Soto, "Observability of the direct product of bilinear systems on Lie groups," *Comput. Math. Appl.*, **36**, No. 3, 107–112 (1998).

18. J. Basto Gonçalves, "Sufficient conditions for local controllability with unbounded controls," *SIAM J. Control Optim.*, **16**, 1371–1378 (1987).

19. J. Basto Gonçalves, "Controllability in codimension one," *J. Diff. Equat.*, **68**, 1–9 (1987).

20. J. Basto Gonçalves, "Local controllability in 3-manifolds," *Syst. Contr. Lett.*, **14**, 45–49 (1990).

21. J. Basto Gonçalves, "Local controllability of scalar input systems on 3-manifolds," *Syst. Contr. Lett.*, **16**, 349–355 (1991).

22. A. Bacciotti, "On the positive orthant controllability of two-dimensional bilinear systems," *Syst. Contr. Lett.*, **3**, 53–55 (1983).

23. A. Bacciotti and G. Stefani, "On the relationship between global and local controllability," *Math. Syst. Theory*, **16**, 79–91 (1983).

24. R. M. Bianchini and G. Stefani, "Sufficient conditions of local controllability," in: Proc. 25th IEEE Conf. Decis. Control, Athens (1986).

25. B. Bonnard, "Contrôllabilité des systèmes bilinéaires," *Math. Syst. Theory*, **15**, 79–92 (1981).

26. B. Bonnard, "Contrôllabilité des systèmes bilinéaires," in: Qutils Modeles Math. Autom. Anal. Syst. Trait Signal., vol. 1, Paris (1981), pp. 229–243.

27. B. Bonnard, "Controllabilité de systèmes mecaniques sur les groupes de Lie," *SIAM J. Control Optim.*, **22**, 711–722 (1984).

28. B. Bonnard, V. Jurdjevic, I. Kupka, and G. Sallet, "Transitivity of families of invariant vector fields on the semidirect products of Lie groups," *Trans. Amer. Math. Soc.*, **271**, No. 2, 525–535 (1982).

29. W. Boothby, "A transitivity problem from control theory," *J. Diff. Equat.*, **17**, 296–307 (1975).

30. W. M. Boothby, "Some comments on positive orthant controllability of bilinear systems," *SIAM J. Control Optim.*, **20**, 634–644 (1982).

31. W. Boothby and E. N. Wilson, "Determination of the transitivity of bilinear systems," *SIAM J. Control*, **17**, 212–221 (1979).

32. A. Borel, "Some remarks about transformation groups transitive on spheres and tori," *Bull. Amer. Math. Soc.*, **55**, 580–586 (1949).

33. A. Borel, "Le plan projectif des octaves et les sphères comme espaces homogènes," *C. R. Acad. Sci. Paris*, **230**, 1378–1380 (1950).

34. N. Bourbaki, *Éléments de Mathématique, Groupes et Algèbres de Lie*, Chapitre 1: Algèbres de Lie, Hermann, Paris (1960).

35. N. Bourbaki, *Éléments de Mathématique, Groupes et Algèbres de Lie*, Chapitre 2: Algèbres de Lie Libres. Chapitre 3: Groupes de Lie, Hermann, Paris (1972).

36. R. W. Brockett, “System theory on group manifolds and coset spaces,” *SIAM J. Control*, **10**, 265–284 (1972).

37. R. W. Brockett, “Lie algebras and Lie groups in control theory,” in: *Geometric Methods in System Theory*, D. Q. Mayne and R. W. Brockett, Eds., *Proceedings of the NATO Advanced study institute held at London, August 27 – September 7*, 1983, Dordrecht – Boston, D. Reidel Publishing Company (1973), pp. 43–82.

38. R. W. Brockett, “On the reachable set for bilinear systems,” *Lect. Notes Econ. and Math. Syst.*, **111**, 54–63 (1975).

39. C. Bruni, G. Di Pillo, and G. Koch, “Bilinear systems: an appealing class of “nearly linear” systems in theory and applications,” *IEEE Trans. Autom. Control*, **19**, 334–348 (1974).

40. J. L. Casti, “Recent developments and future perspectives in nonlinear system theory,” *SIAM Review*, **24**, No. 2, 301–331 (1982).

41. D. Cheng, W. P. Dayawansa, and C. F. Martin, “Observability of systems on Lie groups and coset spaces,” *SIAM J. Control Optim.*, **28**, 570–581 (1990).

42. I. Chong and J. D. Lawson, “Problems on semigroup and control,” *Semigroup Forum*, **41**, 245–252 (1990).

43. P. Crouch and F. Silva Leite, “On the uniform finite generation of $\mathrm{SO}(n;\mathbb{R})$,” *Syst. Contr. Letters*, **2**, 341–347 (1983).

44. P. Crouch and C. I. Byrnes, “Symmetries and local controllability,” in: *Algebraic and Geometric Methods in Nonlinear Control Theory,* M. Fliess and M. Hazewinkel eds., Reidel Publishing Company, Holland, (1986).

45. D. Elliott and T. Tarn, “Controllability and observability for bilinear systems,” in: *SIAM National Meeting*, Seattle, Washington (1971).

46. M. J. Enos, “Controllability of a system of two symmetric rigid bodies in three space,” *SIAM J. Control Optim.*, **32**, 1170–1185 (1994).

47. J. P. Gauthier, *Structures des Systemes Nonlineares*, GNRS, Paris (1984).

48. J. P. Gauthier and G. Bornard, “Contrôlabilité des systèmes bilinèaires,” *SIAM J. Control Optim.*, **20**, No. 3, 377–384 (1982).

49. J. P. Gauthier, I. Kupka and G. Sallet, “Controllability of right invariant systems on real simple Lie groups,” *Syst. Contr. Lett.*, **5**, 187–190 (1984).

50. R. Hermann, *On the Accessibility Problem in Control Theory*, in: International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, pp. 325–332 (1963).

51. H. Hermes, "On local and global controllability," *SIAM J. Control*, **12**, 252–261 (1974).

52. H. Hermes, "Controlled stability," *Ann. Mat. Pura ed Appl.*, **CXIV**, 103–119 (1977).

53. H. Hermes, "On local controllability," *SIAM J. Control Optim.*, **20**, 211–220 (1982).

54. H. Hermes and M. Kawski, "Local controllability of a single-input, affine system," in: *Proc. 7th Int. Conf. Nonlinear Analysis*, Dallas (1986).

55. J. Hilgert, "Maximal semigroups and controllability in products of Lie groups," *Arch. Math.*, **49**, 189–195 (1987).

56. J. Hilgert, "Controllability on real reductive Lie groups," *Math Z.*, **209**, 463–466 (1992).

57. J. Hilgert, K. H. Hofmann and J. D. Lawson, "Controllability of systems on a nilpotent Lie group," *Beiträge Algebra Geometrie*, **20**, 185–190 (1985).

58. J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford University Press (1989).

59. J. Hilgert and K. H. Neeb, "Lie semigroups and their applications," *Lect. Notes Math.*, **1552** (1993).

60. M. W. Hirsch, "Convergence in neural nets," in: *Proc. Int. Conf. Neural Networks*, vol. II (1987), pp. 115–125.

61. R. M. Hirschorn, "Invertibility of control systems on Lie groups," *SIAM J. Contr. Optim.*, **15**, 1034–1049 (1977).

62. K. H. Hofmann, "Lie algebras with subalgebras of codimension one," *Ill. J. Math.*, **9**, 636–643 (1965).

63. K. H. Hofmann, "Hyperplane subalgebras of real Lie algebras," *Geom. Dedic.*, **36**, 207–224 (1990).

64. K. H. Hofmann, "Compact elements in solvable real Lie algebras," *Seminar Sophus Lie* (*Lie theory*), **2**, 41–55 (1992).

65. K. H. Hofmann, "Memo to Yurii Sachkov on Hyperplane Subalgebras of Lie Algebras," E-mail message, March (1996).

66. K. H. Hofmann and J. D. Lawson, "Foundations of Lie semigroups," *Lect. Notes Math.*, **998**, 128–201 (1983).

67. K. R. Hunt, "Controllability of nonlinear hypersurface systems," in: C. I. Byrnes and C. F. Martin Eds., Algebraic and Geometric Methods in Linear Systems Theory, AMS, Providence, Rhode Island (1980).

68. K. R. Hunt, "n-dimensional controllability with $(n-1)$ controls," *IEEE Trans. Autom. Control*, **27**, 113–117 (1982).

69. N. Imbert, M. Clique and A.-J. Fossard, "Un critère de gouvernabilite des systèmes bilinéaires," *R.A.I.R.O.*, **3**, 55–64 (1975).

70. A. Isidori, *Nonlinear Control Systems. An Introduction*, Springer, Berlin (1989).

71. N. Jacobson, *Lie Algebras*, Interscience Publishers, New York and London (1962).

72. I. Joo, N. M. Tuan, "On controllability of some bilinear systems," *C. R. Acad. Sci., Ser. I,* **315**, 1393–1398 (1992).

73. A. Joseph, "The minimal orbit in a simple Lie algebra and its associated maximal ideal," *Ann. Sc. de l'École Normale Sup.*, **9**, No. 1, 1–29 (1976).

74. V. Jurdjevic, "On the reachability properties of curves in $\mathbb{R}^n$ with prescribed curvatures," *University of Bordeaux Publ. 1*, No. 8009 (1980).

75. V. Jurdjevic, "Optimal control problems on Lie groups," in: *Analysis of Controlled Dynamical Systems, Proc. Confer., Lyon, France, July* 1990, B. Bonnard, B. Bride, J. P. Gauthier, I. Kupka Eds., 274–284.

76. V. Jurdjevic, "The geometry of plate-ball problem," *Arch. Rat. Mech. Anal.*, **124**, 305–328 (1993).

77. V. Jurdjevic, "Optimal control problems on Lie groups: Crossroads between geometry and mechanics," in: *Geometry of Feedback and Optimal Control*, ed. B. Jukubczyk and W. Respondek. New York: Marcel Dekker (1993).

78. V. Jurdjevic, "Non-Eucledean elastica," *Amer. J. Math.*, **117**, 93–124 (1995).

79. V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press (1997).

80. V. Jurdjevic and I. Kupka, "Control systems subordinated to a group action: Accessibility," *J. Differ. Equat.*, **39**, 186–211 (1981).

81. V. Jurdjevic and I. Kupka, "Control systems on semi-simple Lie groups and their homogeneous spaces," *Ann. Inst. Fourier*, Grenoble **31**, No. 4, 151–179 (1981).

82. V. Jurdjevic and G. Sallet, "Controllability properties of affine systems," *SIAM J. Control*, **22**, 501–508 (1984).

83. V. Jurdjevic and H. Sussmann, "Control systems on Lie groups," *J. Diff. Equat.*, **12**, 313–329 (1972).

84. M. Kawski, *Nilpotent Lie Algebras of Vector Fields and Local Controllability of Nonlinear Systems*, Ph.D. Dissertation, Univ. of Colorado, Boulder, USA (1986).

85. D. E. Koditschek and K. S. Narendra, "The controllability of planar bilinear systems," *IEEE Trans. Autom. Control*, **30**, 87–89 (1985).

86. A. Krener, "A generalization of Chow's theorem and the Bang-Bang theorem to non-linear control porblems," *SIAM J. Control*, **12**, 43–51 (1974).

87. P. S. Krishnaprasad and D. P. Tsakiris, "G-snakes: Nonholonomic kinematic chains on Lie groups," *Proc. 33rd IEEE Conf. Decis. Control,* Lake Buena Vista, FL (1994), pp. 2955–2960.

88. P. S. Krishnaprasad and D. P. Tsakiris, "Oscillations, $SE(2)$-snakes and motion control," *Proc. 34th IEEE Conf. Decis. and Control*, New Orleans, Louisiana (1995).

89. J. Kučera, "Solution in large of control system $\dot{x} = (A(1-u)+Bu)x$," *Czech. Math. J.*, **16**, 600–623 (1966).

90. J. Kučera, "Solution in large of control system $\dot{x} = (Au + Bv)x$," *Czech. Math. J.*, **17**, 91–96 (1967).

91. J. Kučera, "On the accessibility of bilinear system," *Czech. Math. J.*, **20**, 160–168 (1970).

92. I. Kupka, "Applications of semigroups to geometric control theory," in: *The Analytical and Topological Theory of Semigroups — Trends and Developments*, K. H. Hofmann, J. D. Lawson and J. S. Pym, Eds., de Gruyter Expositions in Mathematics, **1** (1990), pp. 337–345.

93. M. Kuranishi, "On everywhere dense imbedding of free groups in Lie groups," *Nagoya Math. J.*, **2** 63–71 (1951).

94. J. D. Lawson, "Maximal subsemigroups of Lie groups that are total," *Proc. Edinburgh Math. Soc.*, **30**, 479–501 (1985).

95. N. Levitt and H. J. Sussmann, "On controllability by means of two vector fields," *SIAM J. Control,* **13**, 1271–1281 (1975).

96. N. L. Lepe, "Geometric method of investigation of controllability of two-dimensional bilinear systems," *Avtomat. Telemekh.*, No. 11, 19–25 (1984).

97. C. Lobry, "Contrôllabilité des systèmes non linéaires," *SIAM J. Control,* **8**, 573–605 (1970).

98. C. Lobry, "Controllability of non linear systems on compact manifolds," *SIAM J. Control,* **12**, 1–4 (1974).

99. C. Lobry, "Critères de gouvernabilite des asservissements non linéaires," *R.A.I.R.O.,* **10**, 41–54 (1976).

100. L. Markus, "Controllability of multi-trajectories on Lie groups," *Lect. Notes Math.*, **898**, 250–256 (1981).

101. D. Mittenhuber, "Kontrolltheorie auf Lie-Gruppen," *Seminar Sophus Lie*, **1**, 185–191 (1991).

102. D. Mittenhuber, "Semigroups in the simply connected covering of SL(2)," *Semigroup Forum,* **46**, 379–387 (1993).

103. D. Mittenhuber, "Control theory on Lie groups, Lie semigroups and the globality of Lie wedges," Ph. D. Dissertation, TH Darmstadt (1994).

104. F. Monroy-Pérez, "Non-Euclidean Dubins' problem," *J. Dyn. Cont. Syst.*, **4**, 249–272 (1998).

105. D. Montgomery and H. Samelson, "Transformation groups of spheres," *Ann. Math.*, **44**, 454–470 (1943).

106. N. E. Leonard, "Averaging and motion control of systems on Lie groups," Ph.D. dissertation, Univ. Maryland, College Park, MD (1994).

107. N. E. Leonard and P. S. Krishnaprasad, "Control of switched electrical networks using averaging on Lie groups," *Proc.* 33*rd IEEE Conf. Decis. Control,* Lake Buena Vista, FL (1994), pp. 1919–1924.

108. N. E. Leonard and P. S. Krishnaprasad, "Motion control of drift-free, left-invariant systems on Lie groups," *IEEE Trans. Autom. Control*, **40**, 1539–1554 (1995).

109. M. Lovric, "Left-invariant control systems on Lie groups," *Preprint* F193-CT03, January 1993, The Fields Institute for Research in Mathematical Sciences, Canada.

110. K.-H. Neeb, "Semigroups in the universal covering of SL(2)," *Semigroup Forum*, **40**, 33–43 (1990).

111. U. Piechottka, "Comments on 'The controllability of planar bilinear systems'," *IEEE Trans. Autom. Control,* **35**, 767–768 (1990).

112. U. Piechottka and P. M. Frank, "Controllability of bilinear systems: A survey and some new results," in: *Nonlinear Control Systems Design*, ed. A. Isidori, Pergamon Press (1990), pp. 12–28.

113. R. E. Rink and R. R. Möhler, “Completely controllable bilinear systems,” *SIAM J. Control*, **6**, 477–486 (1968).

114. Yu. L. Sachkov, “Controllability of three-dimensional bilinear systems,” Vest. MGU, Mat., Mekh., No. 4, 26–30 (1991).

115. Yu. L. Sachkov, “Invariant domains of three-dimensional bilinear systems,” Vest. MGU, Mat., Mekh., No. 2, 361–363 (1993).

116. Yu. L. Sachkov, “Positive orthant controllability of 2-dimensional and 3-dimensional bilinear systems,” *Diff. Uravn.*, No. 2, 361–363 (1993).

117. Y. L. Sachkov, “Positive orthant controllability of single-input bilinear systems,” *Mat. Zametki*, **85**, 419–424 (1995).

118. Yu. L. Sachkov, “Controllability of hypersurface and solvable invariant systems,” *J. Dyn. Control Syst.*, **2**, No. 1, 55–67 (1996).

119. Yu. L. Sachkov, “On positive orthant controllability of bilinear systems in small codimensions,” *SIAM Journ. Control and Optim.*, **35**, 29–35 (1997).

120. Yu. L. Sachkov, “Controllability of right-invariant systems on solvable Lie groups,” *J. Dyn. Control Syst.*, **3**, No. 4, 531–564 (1997).

121. Yu. L. Sachkov, “On invariant orthants of bilinear systems,” *J. Dyn. Control Syst.*, **4**, No. 1, 137–147 (1998).

122. Yu. L. Sachkov, “Survey on controllability of invariant systems on solvable Lie Groups,” in: Proc. AMS Summer Research Institute on Differential Geometry and Control, Boulder, USA, July 1997, to appear.

123. Yu. L. Sachkov, “Classification of controllability in small-dimensional solvable Lie algebras” (in preparation).

124. G. Sallet, “Une condition suffisante de complète contrôlabilité dans le groupe des déplacements de $\mathbb{R}^n$,” *C. R. Acad. Sc. Paris, Série A*, **282**, 41–44 (1976).

125. G. Sallet, “Complete controllabilite sur les groupes lineaires,” in: *Qutils Modeles Math. Autom. Anal. Syst. Trait Signal.*, Vol. 1, Paris (1981), pp. 215–227.

126. G. Sallet, “Extension techniques,” in: *Systems and Control Encyclopedia II* (1987), pp. 1581–1583.

127. G. Sallet, “Lie groups: Controllability,” in: *Systems and Control Encyclopedia* (1987), pp. 2756–2759.

128. H. Samelson, “Topology of Lie groups,” *Bull. Amer. Math. Soc.*, **58**, 2–37 (1952).

129. L. A. B. San Martin, “Invariant control sets on flag manifolds,” *Math. Control Signals Systems*, **6**, 41–61 (1993).

130. L. A. B. San Martin and P. A. Tonelli, “Semigroup actions on homogeneous spaces,” *Semigroup Forum*, **14**, 1–30 (1994).

131. A. Sarti, G. Walsh, and S. Sastry, “Steering left-invariant control systems on matrix Lie groups,” in: *Proc. 32nd IEEE Conf. Decis. Control*, San Antonio, Texas (1993), pp. 3117–3121.

132. F. Silva Leite, “Uniform controllable sets of left-invariant vector fields on compact Lie groups,” *Syst. Contr. Lett.*, **6**, 329–335 (1986).

133. F. Silva Leite, “Uniform controllable sets of left-invariant vector fields on noncompact Lie Groups,” *Syst. Contr. Lett.*, **7**, 213–216 (1986).

134. F. Silva Leite, “Pairs of generators for compact real forms of the classical Lie algebras,” *Linear Algebra Appl.*, **121**, 123–133 (1989).

135. F. Silva Leite, “Bounds on the order of generation of $\mathrm{SO}(n;\mathbb{R})$ by one-parameter subgroups,” *Rocky Mount. J. Math.*, **21**, 879–911 (1991).

136. F. Silva Leite and P. Crouch, “Controllability on classical Lie groups,” *Math. Control Signals Systems*, **1**, 31–42 (1988).

137. G. Stefani, “On the local controllability of a scalar-input system,” in: *Theory and Applications of Nonlinear Control Systems*, C. I. Byrnes and A. Lindquist Eds., Elsevier Science Publ. (1986).

138. H. J. Sussmann, “The ‘Bang-Bang’ problem for certain control systems on $\mathrm{GL}(n;\mathbb{R})$,” *SIAM J. Control*, **10**, 470–476 (1972).

139. H. J. Sussmann and V. Jurdjevic, “Controllability of non-linear systems,” *J. Diff. Equat.*, **12**, 95–116 (1972).

140. H. J. Sussmann, “Orbits of families of vector fields and integrability of distributions,” *Trans. Amer. Math. Soc.*, **180**, 171–188 (1973).

141. H. J. Sussmann, “Lie brackets, real analyticity and geometric control,” in: *Differential Geometric Control Theory*, R. W Brockett, R. S. Millmann, H. J. Sussmann eds., Birkhäuser, Boston–Basel–Stuttgart (1983), pp. 1–116.

142. H. J. Sussmann, “A general theorem on local controllability,” *SIAM J. Control Optim.*, **25**, 158–194 (1987).

143. A. I. Tretyak, “Sufficient conditions for local controllability and high-order necessary conditions for optimality. A differential-geometric approach,” in:

Progress of Science and Technology, Series on Mathematics and Its Applications. Thematical Surveys, Vol. 24, *Dynamical Systems*–4 [in Russian], All-Russian Institute for Scientific and Technical Information (VINITI), Russian Acad. Nauk (1996).

144. S. A. Vakhrameev, "Geometrical and topological methods in optimal control theory," J. Math. Sci., **76**, No. 5, 2555–2719 (1995).

145. S. A. Vakhrameev and A. V. Sarychev, "Geometrical control theory," in: *Progress of Science and Technology, Series on Algebra, Topology, and Geometry. Thematical Surveys*, Vol. 23 [in Russian], All-Russian Institute for Scientific and Technical Information (VINITI), Russian Acad. Nauk (1985), pp. 197–280.

146. V. S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Spinger-Verlag, New York, Berlin, Heidelberg, Tokyo (1984).

147. E. B. Vinberg and A. L. Onishchik, *Seminar on Lie Groups and Algebraic Groups* [in Russian], Moscow (1988).

148. E. B. Vinberg, V. V. Gorbatcevich, and A. L. Onishchik, "Construction of Lie groups and Lie algebras," in: *Progress of Science and Technology, Series on Complementary Problems in Mathematics, Basic Directions*, Vol. 41, All-Russian Institute for Scientific and Technical Information (VINITI), Russian Acad. Nauk (1989).

149. G. Walsh, R. Montgomery, and S. Sastry, "Optimal path planning on matrix Lie groups," *Proc. 33rd IEEE Conf. Decis. and Control,* Lake Buena Vista, FL (1994), pp. 1312–1318.

150. G. N. Yakovenko, "Optimal control synthesis on a third-order Lie group," *Kybern. Vychisl. Tekhn.* (Kiev), **51**, 17–22 (1981).

151. G. N. Yakovenko, "Control on Lie groups: First integrals, singular controls," *Kybern. Vychisl. Tekhn.* (Kiev), **62**, 10–20 (1984).

152. V. A. Yatcenko, "Euler equation on Lie groups and optimal control of bilinear systems," *Kybern. Vychisl. Tekhn.* (Kiev), **58**, 78–80 (1983).

153. M. I. Zelikin, "Optimal trajectories synthesis on representation spaces of Lie groups," *Mat. Sb.*, **132**, 541–555 (1987).

154. M. I. Zelikin, "Group symmetry in degenerate extremal problems," *Usp. Mat. Nauk*, **43**, No. 2, 139–140 (1988).

155. M. I. Zelikin, "Optimal control of a rigid body rotation," *Dokl. Ross. Akad. Nauk*, **346**, 334–336 (1996).

156. M. I. Zelikin, “Totally extremal manifolds for optimal control problems,” in: *Semigroups in Algebra, Geometry and Analysis*, Eds.: Hofmann, Lawson, Vinberg. Walter de Gruyter & Co., Berlin, New York (1995), pp. 339–354.