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Discrete symmetries in the generalized Dido problem

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Abstract. The generalized Dido problem is considered — a model of the nilpotent sub-Riemannian problem with the growth vector (2, 3, 5). The group of discrete symmetries in this problem is constructed as an extension of the reflection group of the standard mathematical pendulum. The action of these symmetries in the inverse image and image of the exponential map is studied.

Bibliography: 16 titles.

§ 1. Introduction

1.1. Statement of the problem. As is well known, the classical Dido problem can be stated as follows. We are given two points on the plane connected by a curve γ_0, and a number S. It is required to connect these points by a shortest curve γ so that the domain on the plane bounded by the curves γ_0 and γ has algebraic area S. The solution of this problem is an arc of a circle or a segment of a straight line connecting the given points [1], [2].

We consider the following natural generalization of Dido's problem. Suppose that we are given two points (x_0, y_0), $(x_1, y_1) \in \mathbb{R}^2$ connected by some curve $\gamma_0 \subset \mathbb{R}^2$, a number $S \in \mathbb{R}$, and a point $c = (c_x, c_y) \in \mathbb{R}^2$. It is required to find a shortest curve $\gamma \subset \mathbb{R}^2$ connecting the points (x_0, y_0) and (x_1, y_1) such that the domain bounded by the curves γ_0 and γ has the given algebraic area S and centre of mass c.

In [3] it was shown that this problem can be reformulated as an optimal control problem in 5-dimensional space with 2-dimensional control and integral criterion:

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q = (x, y, z, v, w) \in M = \mathbb{R}^5, \quad u = (u_1, u_2) \in U = \mathbb{R}^2,$$
$$q(0) = q_0 = 0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$

where the vector fields for the controls have the form

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2+y^2}{2}\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v}.$$

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From the invariant viewpoint, this is the sub-Riemannian problem

$$\dot q \in \Delta_q,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{\langle \dot q, \dot q \rangle}\, dt \to \min,$$

where

$$\Delta_q = \operatorname{span}(X_1(q), X_2(q)), \qquad q \in M,$$

is a distribution in which the scalar product $\langle \cdot\,, \cdot \rangle$ is defined by the fields X_1, X_2 as an orthonormal basis:

$$\langle X_i, X_j \rangle = \delta_{ij}, \qquad i, j = 1,\, 2.$$

This problem is nilpotent: the fields X_1, X_2 generate a 5-dimensional nilpotent Lie algebra $\operatorname{Lie}(X_1, X_2) = \operatorname{span}(X_1, X_2, X_3, X_4, X_5)$, where

$$X_3 = [X_1, X_2] = \frac{\partial}{\partial z} + x\frac{\partial}{\partial v} + y\frac{\partial}{\partial w}\,,$$
$$X_4 = [X_1, X_3] = \frac{\partial}{\partial v}\,, \qquad X_5 = [X_2, X_3] = \frac{\partial}{\partial w}\,,$$
$$T_qM = \operatorname{span}(X_1, X_2, X_3, X_4, X_5)(q).$$

All the non-trivial commutators in this Lie algebra are exhausted by the following three:

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5,$$

that is, $\operatorname{Lie}(X_1, X_2)$ is a free nilpotent Lie algebra of length 3 with two generators. The flag of the distribution Δ

$$\Delta \subset \Delta^2 = [\Delta, \Delta] \subset \Delta^3 = [\Delta, \Delta^2] \subset \cdots \subset TM,$$

has the form

$$\Delta = L_1 = \operatorname{span}(X_1, X_2),$$
$$\Delta^2 = L_1 \oplus L_2 = \operatorname{span}(X_1, X_2, X_3),$$
$$\Delta^3 = L_1 \oplus L_2 \oplus L_3 = \operatorname{span}(X_1, X_2, X_3, X_4, X_5).$$

Therefore the growth vector of the distribution Δ

$$(n_1, n_2, \dots, n_N), \qquad n_i = \dim \Delta^i(q), \quad n_N = \dim \operatorname{Lie}(\Delta)(q),$$

is equal to (2, 3, 5).

Thus, $(\Delta, \langle \cdot\,, \cdot \rangle)$ is a nilpotent sub-Riemannian structure with the growth vector (2, 3, 5). It is a local quasihomogeneous nilpotent approximation of an arbitrary sub-Riemannian structure on a 5-dimensional manifold with the growth vector (2, 3, 5) (see [4], [5], as well as [6]). As shown in [7], such a nilpotent structure is unique. The generalized Dido problem is a model of the nilpotent sub-Riemannian problem with the growth vector (2, 3, 5).

1.2. Known results. The present paper is a continuation of [3], [7]; we shall constantly use the results of these papers.

In [3] we proved the existence of optimal controls in the generalized Dido problem. By using the Pontryagin maximum principle in the invariant form [8] we constructed a Hamiltonian system for the normal extremals and found the abnormal extremals. We calculated the continuous symmetries of the problem in [7]. By using them, in [3] we showed that the exponential map is factorized by the action of a two-parameter symmetry group (of rotations and dilatations). The normal Hamiltonian system was integrated in terms of the Jacobian elliptic functions. The abnormal geodesics are optimal up to infinity. Small arcs of normal geodesics are optimal, but large arcs, generally speaking, are not; the points where a geodesic ceases to be optimal are called *cut points*.

1.3. Contents of the paper. In this paper we begin the search for the cut points in the generalized Dido problem. It is known that normal geodesics may cease to be optimal for two reasons: either different geodesics of equal length intersect at a given point (Maxwell points), or a family of geodesics has an envelope (conjugate points). For problems with a rich symmetry group the Maxwell points can be sought as the fixed points of the composite of the exponential map and symmetries: if a symmetry permutes geodesics but fixes their common end-point, then this end-point is a Maxwell point.

In the present paper we construct the group of discrete symmetries of the exponential map in the generalized Dido problem. This is a dihedral group, which arises as a result of the existence of reflections in the phase plane of the standard pendulum. After reduction by the two-dimensional group of continuous symmetries, the vertical part of the Hamiltonian system of the Pontryagin maximum principle becomes the system of the standard pendulum. We extend the reflections of the standard pendulum to reflections in the inverse image and image of the exponential map. These discrete symmetries have a simple geometric meaning for the Euler elastics — the projections of geodesics onto the plane (x, y): the reflections of elastics in the centre of a chord, in the chord itself, and in the perpendicular bisector of the chord. Discrete symmetries are factorized by the action of rotations and dilatations. The action of reflections has an especially simple representation in the special elliptic coordinates generated by the phase flow of the standard pendulum.

The procedure for extending the symmetries of the standard pendulum to symmetries of the exponential map is of a general nature and can be applied to a number of optimal control problems in which an independent subsystem of the Hamiltonian system of the Pontryagin maximum principle has a non-trivial symmetry group; this category includes, for example, the well-known problem of a sphere rolling on a plane [9].

The description of discrete symmetries of the exponential map in the generalized Dido problem obtained in the present paper will be used for a complete description of the Maxwell points corresponding to these symmetries and for finding certain conjugate points along geodesics. Thus, an upper estimate will be obtained for the cut time on all the geodesics. These results will be expounded in the subsequent papers [10], [11].

We used the system “Mathematica” [12] to carry out complicated calculations and to produce illustrations in this paper.

The author is grateful to A. A. Agrachev for posing the problem and for useful discussions during the work.

§ 2. Reflections

2.1. Reflections of the field of directions of the standard pendulum. As is well known [3], for the generalized Dido problem the vertical part of the normal Hamiltonian system of the Pontryagin maximum principle reduces (after factorization by two symmetries) to the standard pendulum equation

$$\begin{cases} \dot\theta = c, & \theta \in S^1; \\ \dot c = -\sin\theta, & c \in \mathbb{R}. \end{cases} \tag{1}$$

It is easy to see that the following reflections of the cylinder $S^1 \times \mathbb{R}$ preserve the field of directions of the pendulum (see Fig. 1):

$$\begin{aligned} &\varepsilon^1\colon (\theta, c) \mapsto (\theta, -c), \\ &\varepsilon^2\colon (\theta, c) \mapsto (-\theta, c), \\ &\varepsilon^3\colon (\theta, c) \mapsto (-\theta, -c). \end{aligned}$$

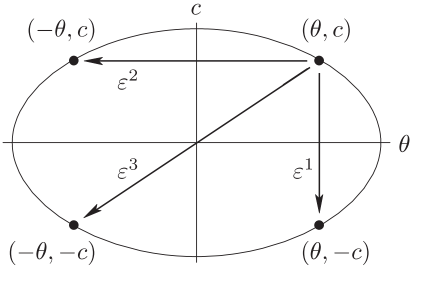


Figure 1. Reflections in the phase plane of the pendulum

The reflections ε^1 and ε^2 change the direction of time, and ε^3 preserves the direction of time on the trajectories of the pendulum. These reflections generate the dihedral group (the symmetry group of a rectangle)

$$D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$$

with the multiplication table

	ε^1	ε^2	ε^3
ε^1	Id	ε^3	ε^2
ε^2	ε^3	Id	ε^1
ε^3	ε^2	ε^1	Id

Then we extend step-by-step the action of the reflections ε^i so that they become symmetries of the exponential map: $\mathrm{Exp} \circ \varepsilon^i = \varepsilon^i \circ \mathrm{Exp}$ (see Proposition 2.6).

2.2. Reflections of the trajectories of the standard pendulum. The action of the symmetries ε^i can be extended to the set of trajectories of the pendulum equation (preserving the direction of time). Let

$$\gamma = \{(\theta_s, c_s) \mid s \in [0,t]\}$$

be a smooth curve on the phase cylinder of the standard pendulum $S^1 \times \mathbb{R}$. We define the maps of curves as follows:

$$\begin{aligned}
\varepsilon^1 \colon \gamma \mapsto \gamma^1 &= \{(\theta_s^1, c_s^1) \mid s \in [0,t]\} = \{(\theta_{t-s}, -c_{t-s}) \mid s \in [0,t]\},\\
\varepsilon^2 \colon \gamma \mapsto \gamma^2 &= \{(\theta_s^2, c_s^2) \mid s \in [0,t]\} = \{(-\theta_{t-s}, c_{t-s}) \mid s \in [0,t]\},\\
\varepsilon^3 \colon \gamma \mapsto \gamma^3 &= \{(\theta_s^3, c_s^3) \mid s \in [0,t]\} = \{(-\theta_s, -c_s) \mid s \in [0,t]\}.
\end{aligned}$$

Proposition 2.1. *The group of reflections $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ preserves the family of trajectories of the standard pendulum* (1).

Proof. The proposition is proved by straightforward differentiation. For example, for ε^1 we obtain

$$\begin{aligned}
\frac{d}{ds}\theta_s^1 &= \frac{d}{ds}\theta_{t-s} = -\dot\theta_{t-s} = -c_{t-s} = c_s^1,\\
\frac{d}{ds}c_s^1 &= \frac{d}{ds}(-c_{t-s}) = \dot c_{t-s} = -\sin\theta_{t-s} = -\sin\theta_s^1.
\end{aligned}$$

Thus, if a curve γ is a trajectory of the pendulum, then the curves γ^1, γ^2, γ^3 are also trajectories of the pendulum (see Fig. 2).

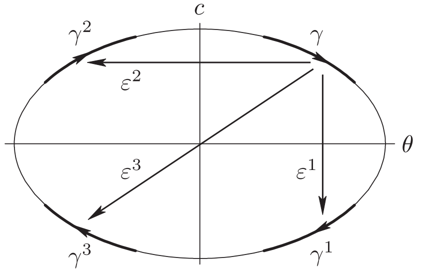


Figure 2. Reflections of the trajectories of the pendulum

2.3. Reflections of the trajectories of the generalized pendulum. In [3] it was shown that the vertical part of the normal Hamiltonian system of the Pontryagin maximum principle for the generalized Dido problem is the system of equations of a generalized pendulum

$$\begin{cases}
\dot\theta = c, & \theta \in S^1;\\
\dot c = -\alpha\sin(\theta-\beta), & c \in \mathbb{R};\\
\dot\alpha = 0, & \alpha \geqslant 0;\\
\dot\beta = 0, & \beta \in S^1.
\end{cases} \tag{2}$$

We consider smooth curves of the form

$$\gamma = \{(\theta_s, c_s, \alpha, \beta) \mid s \in [0,t],\ \alpha, \beta = \mathrm{const}\}$$

in the phase space of the generalized pendulum. We extend the action of the reflections ε^i to the family of such curves as follows:

$$\begin{aligned}
\varepsilon^1\colon \gamma \mapsto \gamma^1 &= \{(\theta_s^1, c_s^1, \alpha, \beta^1)\} = \{(\theta_{t-s}, -c_{t-s}, \alpha, \beta)\},\\
\varepsilon^2\colon \gamma \mapsto \gamma^2 &= \{(\theta_s^2, c_s^2, \alpha, \beta^2)\} = \{(-\theta_{t-s}, c_{t-s}, \alpha, -\beta)\},\\
\varepsilon^3\colon \gamma \mapsto \gamma^3 &= \{(\theta_s^3, c_s^3, \alpha, \beta^3)\} = \{(-\theta_s, -c_s, \alpha, -\beta)\}.
\end{aligned}$$

Proposition 2.2. *The group of reflections $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ preserves the family of trajectories of the generalized pendulum* (2).

Proof. The proof reduces to differentiation in the same way as for Proposition 2.1.

2.4. Reflections of normal extremals. As shown in [3], the normal Hamiltonian system of the Pontryagin maximum principle has triangular form under the parametrization $\lambda = (h_1, \dots, h_5; q) \in T^*M$: the subsystem for the coordinates $(h_1, \dots, h_5) \in T_q^*M$ in a fibre of the cotangent bundle is independent of the point $q \in M$ (recall that $h_i(\lambda) = \langle \lambda, X_i(q) \rangle$). This independent vertical subsystem on the level surface of the Hamiltonian $H = (h_1^2 + h_2^2)/2 = 1/2$ is written in the coordinates $(\theta, c, \alpha, \beta) \in T_q^*M$ as the generalized pendulum (2), where $h_1 = \cos\theta$, $h_2 = \sin\theta$, $h_3 = c$, $h_4 = \alpha \sin\beta$, $h_5 = -\alpha\cos\beta$. We extend the action of the reflections ε^i from the vertical subsystem to the complete Hamiltonian system.

Let $\nu = (\lambda, t) \in N = C \times \mathbb{R}_+$ be a point in the inverse image of the exponential map

$$\mathrm{Exp}\colon N \mapsto M, \qquad \mathrm{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda) = q_t,$$

where $C = \{H = 1/2\} \cap T_{q_0}^*M$ is the initial cylinder for the extremals. Then $\lambda_s = e^{s\vec{H}}(\lambda)$, $s \in [0,t]$, is the corresponding normal extremal, $\mathrm{Exp}(\nu) = \pi(\lambda_t) = q_t$. Henceforth, $e^{s\vec{H}}$ denotes the flow of the Hamiltonian field $\vec{H}$ with Hamiltonian H. We write down the extremal in coordinates as $\lambda_s = (\theta_s, c_s, \alpha, \beta; q_s)$. Then the normal Hamiltonian system of the Pontryagin maximum principle takes the form

$$\dot\lambda_s = \vec{H}(\lambda_s): \qquad \begin{cases} \dot\theta_s = c_s;\\ \dot c_s = -\alpha \sin(\theta_s - \beta);\\ \dot\alpha = 0;\\ \dot\beta = 0;\\ \dot q_s = \cos\theta_s\ X_1(q_s) + \sin\theta_s\ X_2(q_s). \end{cases} \tag{3}$$

In § 2.3 we defined the action of reflections on a trajectory of the vertical subsystem:

$$\varepsilon^i\colon \{(\theta_s, c_s, \alpha, \beta)\} \mapsto \{(\theta_s^i, c_s^i, \alpha, \beta^i)\}, \qquad i = 1,\ 2,\ 3, \quad s \in [0,t].$$

We extend this action to the trajectories of the normal Hamiltonian system (3):

$$\begin{gathered}
\varepsilon^i\colon \{\lambda_s\} = \{(\theta_s, c_s, \alpha, \beta; q_s)\} \mapsto \{\lambda_s^i\} = \{(\theta_s^i, c_s^i, \alpha, \beta^i; q_s^i)\},\\
i = 1,\ 2,\ 3, \qquad s \in [0,t],
\end{gathered}$$

where the curve q_s^i is the solution of the Cauchy problem for the reflected controls

$$\dot{q}_s^i = \cos\theta_s^i\ X_1(q_s^i) + \sin\theta_s^i\ X_2(q_s^i), \qquad s \in [0,t],$$
$$q_0^i = q_0.$$

Clearly, the following assertion holds.

Proposition 2.3. *The group of reflections* $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ *preserves the family of trajectories of the normal Hamiltonian system* (3).

The action of reflections on the geodesics q_s will be described in §§ 2.6, 2.7.

2.5. Reflections in the inverse image of the exponential map. In § 2.4 we defined the action of the reflections ε^i on the normal extremals $\lambda_s = e^{s\vec{H}}(\lambda)$

$$\varepsilon^i \colon \lambda_s \mapsto \lambda_s^i, \qquad s \in [0,t]. \tag{4}$$

Our aim is to represent the reflections as symmetries of the exponential map $\mathrm{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda)$.

We now define the action of reflections in the inverse image of the exponential map. We set

$$\varepsilon^i \colon C \times \mathbb{R}_+ \to C \times \mathbb{R}_+, \qquad (\lambda, t) \mapsto (\lambda^i, t),$$

where $\lambda^i = \lambda_0^i$ is the initial point of the corresponding reflected extremal (4). In coordinates we have

$$\varepsilon^1 \colon (\theta, c, \alpha, \beta, t) \mapsto (\theta^1, c^1, \alpha, \beta^1, t) = (\theta_t, -c_t, \alpha, \beta, t), \tag{5}$$
$$\varepsilon^2 \colon (\theta, c, \alpha, \beta, t) \mapsto (\theta^2, c^2, \alpha, \beta^2, t) = (-\theta_t, c_t, \alpha, -\beta, t), \tag{6}$$
$$\varepsilon^3 \colon (\theta, c, \alpha, \beta, t) \mapsto (\theta^3, c^3, \alpha, \beta^3, t) = (-\theta, -c, \alpha, -\beta, t). \tag{7}$$

We should now like to define the action of reflections in the image of the exponential map as

$$\varepsilon^i \colon M \to M, \qquad q_t \mapsto q_t^i.$$

But it is not clear a priori that the point q_t^i is uniquely determined by the point q_t. To verify this fact we examine the action of reflections on geodesics.

2.6. Reflections of Euler elastics. System (3) shows that the projections of the normal geodesics q_s onto the plane (x, y) satisfy the differential equations

$$\dot{x}_s = \cos\theta_s,$$
$$\dot{y}_s = \sin\theta_s,$$

where the angle θ_s is in turn a solution of the pendulum equation

$$\ddot{\theta}_s = -\alpha\sin(\theta_s - \beta), \qquad \alpha, \beta = \mathrm{const}.$$

Such curves (x_s, y_s) are called *Euler elastics*; they were discovered by Euler as the stationary profiles of an elastic rod. The elastics are the extremals of the functional $\dfrac{1}{2}\displaystyle\int_\gamma \varkappa^2(s)\,ds$ for planar curves, where $\varkappa$ is the curvature of a curve γ

(see, for example, [9], [13]). The Euler elastics (the projections of solutions of the generalized Dido problem) form, up to rotations and dilatations, a one-parameter family of curves connecting a straight line and a circle (the projections of solutions of the classical Dido problem). Sketches of various types of elastics were given in [3].

Let $q_s = (x_s, y_s, z_s, v_s, w_s)$, $s \in [0,t]$, be a geodesic, and let

$$q_s^i = (x_s^i, y_s^i, z_s^i, v_s^i, w_s^i), \qquad s \in [0,t], \quad i = 1,\ 2,\ 3,$$

be its images under the action of reflections (see § 2.4). The curves (x_s, y_s) and (x_s^i, y_s^i) are Euler elastics. The action of the reflections ε^i on these curves is described in the following proposition.

Proposition 2.4. *We have*

1) $(x_s^1, y_s^1) = (x_t - x_{t-s}, y_t - y_{t-s})$, $s \in [0,t]$;
2) $(x_s^2, y_s^2) = (x_t - x_{t-s}, y_{t-s} - y_t)$, $s \in [0,t]$;
3) $(x_s^3, y_s^3) = (x_s, -y_s)$, $s \in [0,t]$.

Proof. We consider only the action of ε^1; the reflections ε^2, ε^3 can be examined in similar fashion. We have

$$x_s^1 = \int_0^s \cos\theta_r^1\,dr = \int_0^s \cos\theta_{t-r}\,dr = \int_{t-s}^t \cos\theta_r\,dr.$$

Taking into account that

$$x_{t-s} = \int_0^{t-s} \cos\theta_r\,dr,$$

we obtain

$$x_s^1 + x_{t-s} = \int_0^t \cos\theta_r\,dr = x_t,$$

that is,

$$x_s^1 = x_t - x_{t-s}, \qquad s \in [0,t].$$

Similarly we obtain

$$y_s^1 = y_t - y_{t-s}, \qquad s \in [0,t].$$

Remark. We note the graphic meaning of the action of reflections on elastics $\{(x_s, y_s) \mid s \in [0,t]\}$ in the case $(x_t, y_t) \neq (x_0, y_0)$.

By the equality

$$\varepsilon^1 : \begin{pmatrix} x_s \\ y_s \end{pmatrix} \overset{\text{(a)}}{\mapsto} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \overset{\text{(b)}}{\mapsto} \begin{pmatrix} x_t - x_{t-s} \\ y_t - y_{t-s} \end{pmatrix} = \begin{pmatrix} x_s^1 \\ y_s^1 \end{pmatrix}$$

the reflection ε^1 is the composite of two transformations: (a) the inversion of time on the elastic, and (b) the reflection of the plane (x, y) in the centre $p_c = (x_t/2, y_t/2)$ of the chord of the elastic, that is, of the straight segment connecting its initial point $(x_0, y_0) = (0,0)$ and end-point (x_t, y_t) (see Fig. 3).

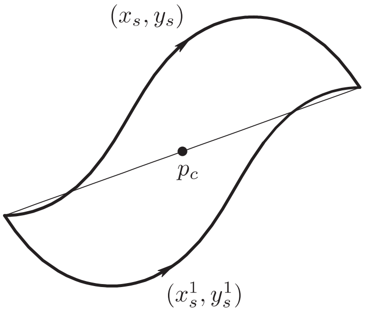


Figure 3. The reflection ε^1 of an elastic in the centre of the chord p_c

For the reflection ε^2 we have the decomposition

$$\begin{aligned}\varepsilon^2 \colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} &\overset{(a)}{\mapsto} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \overset{(b)}{\mapsto} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} -\cos 2\chi & -\sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \\ &\overset{(c)}{\mapsto} \begin{pmatrix} \cos 2\chi & \sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \left[\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} -\cos 2\chi & -\sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \right] \\ &= \begin{pmatrix} x_t - x_{t-s} \\ y_{t-s} - y_t \end{pmatrix} = \begin{pmatrix} x_s^2 \\ y_s^2 \end{pmatrix},\end{aligned}$$

where χ is the polar angle of the point (x_t, y_t):

$$\cos \chi = \frac{x_t}{\sqrt{x_t^2 + y_t^2}}, \qquad \sin \chi = \frac{y_t}{\sqrt{x_t^2 + y_t^2}}.$$

In other words, the reflection ε^2 acts on elastics as the composite of three transformations: (a) the inversion of time on the elastic, (b) the reflection of the plane (x, y) in the perpendicular bisector $l^\perp$ of the chord, and (c) the rotation through the angle (-2χ) (see Fig. 4).

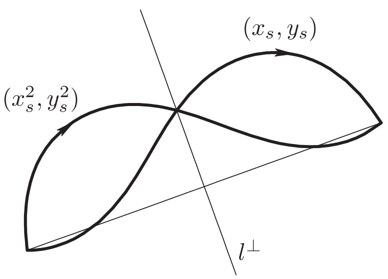


Figure 4. The reflection ε^2 of an elastic in the perpendicular bisector $l^\perp$

The reflection ε^3 is represented as follows:

$$\begin{aligned}\varepsilon^3\colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} &\overset{\text{(a)}}{\mapsto} \begin{pmatrix} \cos 2\chi & \sin 2\chi \\ \sin 2\chi & -\cos 2\chi \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix} \\ &\overset{\text{(b)}}{\mapsto} \begin{pmatrix} \cos 2\chi & \sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \left[\begin{pmatrix} \cos 2\chi & \sin 2\chi \\ \sin 2\chi & -\cos 2\chi \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix} \right] \\ &= \begin{pmatrix} x_s \\ -y_s \end{pmatrix} = \begin{pmatrix} x_s^3 \\ y_s^3 \end{pmatrix};\end{aligned}$$

this is the composite of (a) the reflection of the plane (x, y) in the chord of the elastic, and (b) the rotation through the angle (-2χ), that is, the reflection in the chord l (see Fig. 5).

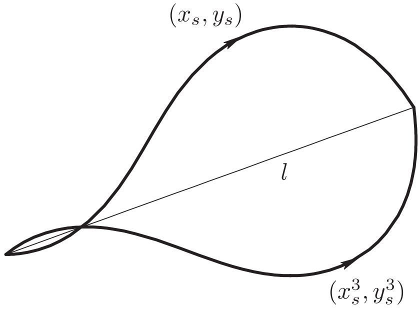


Figure 5. The reflection ε^3 of an elastic in the chord l

Thus, modulo the inversion of time on elastics and rotations of the plane (x, y), we have

i) ε^1 is the reflection of an elastic in the centre of the chord;
ii) ε^2 is the reflection of an elastic in the perpendicular bisector of the chord;
iii) ε^3 is the reflection of an elastic in the chord.

2.7. Reflections of the end-points of geodesics. We now describe the action of the reflections ε^i on the end-points of geodesics $q_t = (x_t, y_t, z_t, v_t, w_t)$.

Proposition 2.5. *We have*

1) $(x_t^1, y_t^1, z_t^1, v_t^1, w_t^1) = (x_t, y_t, -z_t, v_t - x_t z_t, w_t - y_t z_t)$;
2) $(x_t^2, y_t^2, z_t^2, v_t^2, w_t^2) = (x_t, -y_t, z_t, -v_t + x_t z_t, w_t - y_t z_t)$;
3) $(x_t^3, y_t^3, z_t^3, v_t^3, w_t^3) = (x_t, -y_t, -z_t, -v_t, w_t)$.

Proof. We shall prove equality 1); the other two equalities can be considered in similar fashion. The equality $(x_t^1, y_t^1) = (x_t, y_t)$ follows from Proposition 2.4. Next, taking into account the same proposition and the equalities $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$, $\dot{z} = (x\dot{y} - \dot{x}y)/2$, by virtue of the normal Hamiltonian system (3) in the coordinates

(x, y, z, v, w) we obtain

$$\begin{aligned}
z_t^1 &= \frac{1}{2}\int_0^t \bigl(-\cos\theta_{t-s}(y_t - y_{t-s}) + \sin\theta_{t-s}(x_t - x_{t-s})\bigr)\,ds \\
&= \frac{1}{2}\int_0^t (-\cos\theta_{t-s} y_t + \sin\theta_{t-s} x_t)\,ds \\
&\qquad - \frac{1}{2}\int_0^t (-\cos\theta_{t-s} y_{t-s} + \sin\theta_{t-s} x_{t-s})\,ds \\
&= -\frac{1}{2}x_t y_t + \frac{1}{2}y_t x_t + \frac{1}{2}\int_t^0 (-\cos\theta_s y_s + \sin\theta_s x_s)\,ds \\
&= -z_t.
\end{aligned}$$

The equalities $v_t^1 = v_t - x_t z_t$ and $w_t^1 = w_t - y_t z_t$ can be proved in similar fashion.

Proposition 2.5 shows that the end-point of the reflected geodesic q_t^i is uniquely determined by the end-point of the original geodesic q_t. Therefore we can define the action of reflections on the end-points of geodesics as the maps

$$\varepsilon^i \colon M \to M, \qquad \varepsilon^i \colon q_t \to q_t^i.$$

Proposition 2.5 describes the action of this map in the coordinates (x, y, z, v, w) on M.

Remark. The reflection

$$\varepsilon^3 \colon (x, y, z, v, w) \mapsto (x, -y, -z, -v, w)$$

is a symmetry of the nilpotent sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$ defined by the fields X_1, X_2 as an orthonormal basis. This reflection acts on the basis as

$$\varepsilon^3_* \colon (X_1, X_2) \mapsto (X_1, -X_2),$$

that is, this is the reflection of the plane Δ in the straight line $\mathbb{R}X_1$.

2.8. Reflections as symmetries of the exponential map. The definition of the action of the reflections ε^i in the inverse image and image of the exponential map implies that the reflections are symmetries of the exponential map. We thus obtain the following.

Proposition 2.6. *The following diagrams are commutative*:

$$\begin{array}{ccccccc}
C \times \mathbb{R}_+ & \xrightarrow{\text{Exp}} & M & \qquad & (\lambda_0, t) & \xrightarrow{\text{Exp}} & q_t \\
\downarrow{\scriptstyle \varepsilon^i} & & \downarrow{\scriptstyle \varepsilon^i} & & \downarrow{\scriptstyle \varepsilon^i} & & \downarrow{\scriptstyle \varepsilon^i} \\
C \times \mathbb{R}_+ & \xrightarrow{\text{Exp}} & M, & & (\lambda_0^i, t) & \xrightarrow{\text{Exp}} & q_t^i,
\end{array}$$

that is,

$$\varepsilon^i \circ \operatorname{Exp} = \operatorname{Exp} \circ \varepsilon^i, \qquad i = 1,\ 2,\ 3.$$

§ 3. The symmetry group of the exponential map

Along with the discrete symmetry group $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$, the exponential map has the continuous two-parameter symmetry group $e^{\mathbb{R}\vec{h}_0} \circ e^{\mathbb{R}Z}$, where

$$h_0(\lambda) = \langle \lambda, X_0(q) \rangle, \qquad X_0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w},$$

$$Z = \vec{h}_Y + e, \qquad h_Y(\lambda) = \langle \lambda, Y(q) \rangle,$$

$$Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 3v\frac{\partial}{\partial v} + 3w\frac{\partial}{\partial w}, \qquad e = \sum_{i=1}^{5} h_i \frac{\partial}{\partial h_i}.$$

Indeed, according to Assertion 4.2 in [3],

$$e^{s\vec{h}_0} \circ e^{rZ} \circ e^{t\vec{H}}(\lambda) = e^{t'\vec{H}} \circ e^{s\vec{h}_0} \circ e^{rZ}(\lambda), \qquad t' = te^r,$$

whence by using the canonical projection $\pi \colon T^*M \to M$, $\pi(\lambda) = q$, we obtain

$$e^{sX_0} \circ e^{rY} \circ \mathrm{Exp}(\lambda, t) = \mathrm{Exp}(e^{s\vec{h}_0} \circ e^{rZ}(\lambda), t'). \tag{8}$$

We define the natural action of continuous symmetries in the inverse image of the exponential map:

$$e^{s\vec{h}_0} \colon C \times \mathbb{R}_+ \to C \times \mathbb{R}_+, \qquad (\lambda, t) \mapsto (e^{s\vec{h}_0}(\lambda), t),$$
$$e^{rZ} \colon C \times \mathbb{R}_+ \to C \times \mathbb{R}_+, \qquad (\lambda, t) \mapsto (e^{rZ}(\lambda), te^r).$$

From this definition and equality (8) we obtain the following.

Proposition 3.1. *The following diagrams are commutative*:

$$\begin{array}{ccc} C \times \mathbb{R}_+ & \xrightarrow{\mathrm{Exp}} & M \\ \big\downarrow e^{s\vec{h}_0} & & \big\downarrow e^{sX_0} \\ C \times \mathbb{R}_+ & \xrightarrow{\mathrm{Exp}} & M, \end{array} \qquad \begin{array}{ccc} C \times \mathbb{R}_+ & \xrightarrow{\mathrm{Exp}} & M \\ \big\downarrow e^{rZ} & & \big\downarrow e^{rY} \\ C \times \mathbb{R}_+ & \xrightarrow{\mathrm{Exp}} & M, \end{array}$$

that is,

$$e^{sX_0} \circ \mathrm{Exp} = \mathrm{Exp} \circ e^{s\vec{h}_0}, \qquad e^{rY} \circ \mathrm{Exp} = \mathrm{Exp} \circ e^{rZ}.$$

Thus, the group generated by reflections, rotations, and dilatations

$$G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3, e^{s\vec{h}_0}, e^{rZ} \rangle$$

preserves the exponential map. We now find the commutation relations in the group G. The flows of the fields $\vec{h}_0$ and Z commute with each other [3]. The reflections ε^i also commute with each other (see the multiplication table in § 2.1). We now determine the rules of commutation of discrete and continuous symmetries.

Proposition 3.2. *In the inverse image of the exponential map* $C \times \mathbb{R}_+$ *the following commutation relations hold*:

$$e^{s\vec{h}_0} \circ \varepsilon^1 = \varepsilon^1 \circ e^{s\vec{h}_0}, \qquad e^{s\vec{h}_0} \circ \varepsilon^2 = \varepsilon^2 \circ e^{-s\vec{h}_0}, \qquad e^{s\vec{h}_0} \circ \varepsilon^3 = \varepsilon^3 \circ e^{s\vec{h}_0},$$
$$e^{rZ} \circ \varepsilon^i = \varepsilon^i \circ e^{rZ}, \qquad i = 1, 2, 3.$$

Proof. The proposition can be proved by a straightforward calculation using the expressions for the action of reflections (5)–(7) and the continuous symmetries

$$e^{s\vec{h}_0}\colon (\theta, c, \alpha, \beta, t) \mapsto (\theta + s, c, \alpha, \beta + s, t),$$
$$e^{rZ}\colon (\theta, c, \alpha, \beta, t) \mapsto (\theta, ce^{-r}, \alpha e^{-2r}, \beta, te^{r})$$

in the inverse image of the exponential map.

Corollary 3.1. *In the image of the exponential map M the following commutation relations hold*:

$$e^{sX_0} \circ \varepsilon^1 = \varepsilon^1 \circ e^{sX_0}, \qquad e^{sX_0} \circ \varepsilon^2 = \varepsilon^2 \circ e^{-sX_0}, \qquad e^{sX_0} \circ \varepsilon^3 = \varepsilon^3 \circ e^{sX_0},$$
$$e^{rY} \circ \varepsilon^i = \varepsilon^i \circ e^{rY}, \qquad i = 1,\ 2,\ 3.$$

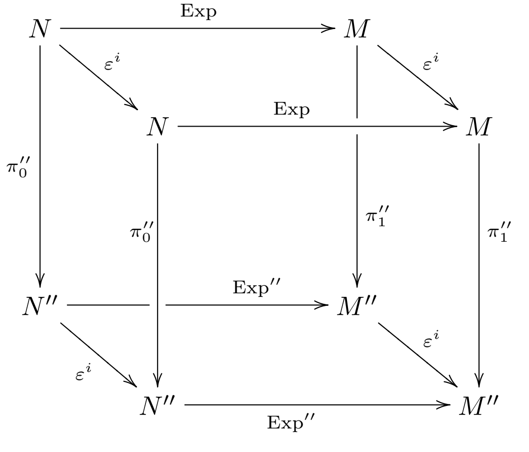
Remark. We denote the inverse image of the exponential map by $N = C \times \mathbb{R}_+$. We consider the quotient spaces of the inverse image and image of the exponential map, as well as the corresponding canonical projections:

$$N'' = N/G_{\vec{h}_0,Z}, \qquad \pi_0''\colon N \to N'',$$
$$M'' = M/G_{X_0,Y}, \qquad \pi_1''\colon M \to M'',$$

where $G_{\vec{h}_0,Z} = \langle e^{s\vec{h}_0}, e^{rZ}\rangle$ and $G_{X_0,Y} = \langle e^{sX_0}, e^{rY}\rangle$ are two-parameter groups of continuous symmetries in N and M, respectively; see [3]. Proposition 3.2 and Corollary 3.1 allow us to define the action of reflections in the quotient spaces

$$\varepsilon^i\colon N'' \to N'', \qquad \varepsilon^i\colon M'' \to M''$$

so that the following diagram is commutative:



§ 4. Action of reflections in the inverse image of the exponential map

In [3] the so-called elliptic coordinates were introduced in subdomains of the phase space of the pendulum. These coordinates are generated by the "action–angle" coordinates for the standard pendulum. One of these coordinates is time on the trajectories of the pendulum, and the second is the reparametrized energy of the pendulum. In this section we recall the construction of elliptic coordinates (with certain corrections and additions); then using these coordinates we describe the action of reflections in the inverse image of the exponential map.

The twenty-sixth of the "Lectures on dynamics" of C. Jacobi [14] is called "Elliptic coordinates" and begins with the well-known words:

> "The main difficulty in the integration of these differential equations is the introduction of convenient variables, there being no general rule for finding them. Therefore one has to adopt the opposite approach and, after finding a remarkable substitution, to seek the problems for which this substitution can be successfully used".

Note that the coordinates introduced below are unrelated to Jacobi's elliptic coordinates. Moreover, our procedure was opposite to that described by Jacobi: we had to introduce our elliptic coordinates specifically for parametrizing the sub-Riemannian geodesics [3] and finding the Maxwell points [10], [11]. Elliptic coordinates lift the veil of complexity over the problems governed by the pendulum equation and open their solution to our eyes (see Fig. 6). Here we have an important intersection point with Jacobi: our coordinates are introduced by using the Jacobian elliptic functions [15], [16]. Another important moment will be the study of the conjugate points, that is, the solutions of the Jacobi equation, along geodesics [11].

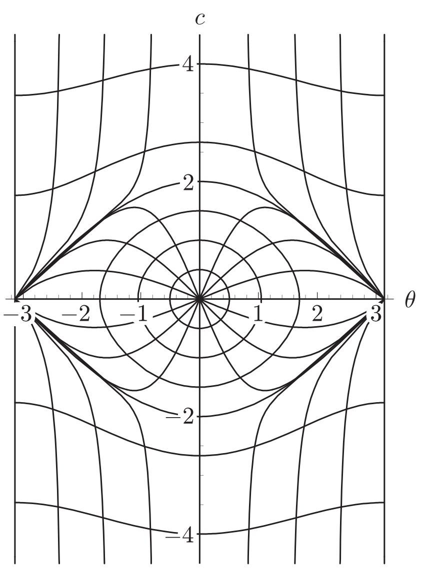


Figure 6. Grid of elliptic coordinates

4.1. Elliptic coordinates in the initial cylinder. Recall [3] that the set of initial points of the extremals is a cylinder — the phase space of the generalized pendulum (2)

$$C = T^*_{q_0} M \cap H^{-1}\left(\frac{1}{2}\right) = \{(\theta \in S^1, c \in \mathbb{R}, \alpha \geqslant 0, \beta \in S^1)\},$$

and the energy of the generalized pendulum is equal to

$$E = \frac{c^2}{2} - \alpha \cos(\theta - \beta) \in [-\alpha, +\infty).$$

In [3] we introduced the partition of the cylinder C into the subsets

$$\begin{aligned}
&C = \bigcup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \varnothing, \quad i \neq j,\\
&C_1 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (-\alpha, \alpha)\},\\
&C_2 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (\alpha, +\infty)\},\\
&C_3 = \{\lambda \in C \mid \alpha \neq 0,\ E = \alpha,\ \theta - \beta \neq \pi\},\\
&C_4 = \{\lambda \in C \mid \alpha \neq 0,\ E = -\alpha\},\\
&C_5 = \{\lambda \in C \mid \alpha \neq 0,\ E = \alpha,\ \theta - \beta = \pi\},\\
&C_6 = \{\lambda \in C \mid \alpha = 0,\ c \neq 0\},\\
&C_7 = \{\lambda \in C \mid \alpha = c = 0\}.
\end{aligned}$$

In the subsets C_1, C_2, C_3 the elliptic coordinates $(k, \varphi, \alpha, \beta)$ are introduced as follows:

$$C_1 = \left\{k \in (0,1),\ \varphi\left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right),\ \alpha > 0,\ \beta(\operatorname{mod} 2\pi)\right\},$$

$$k = \sqrt{\frac{E+\alpha}{2\alpha}} = \sqrt{\sin^2 \frac{\theta - \beta}{2} + \frac{c^2}{4\alpha}} \in (0,1),$$

$$\begin{cases} \sin \dfrac{\theta - \beta}{2} = s_1 k \operatorname{sn}(\sqrt{\alpha}\,\varphi); \\ \dfrac{c}{2} = k\sqrt{\alpha} \operatorname{cn}(\sqrt{\alpha}\,\varphi), \end{cases} \qquad s_1 = \operatorname{sgn} \cos \frac{\theta - \beta}{2}.$$

In this case,

$$\cos \frac{\theta - \beta}{2} = s_1 \operatorname{dn}(\sqrt{\alpha}\,\varphi).$$

Henceforth we use the Jacobian elliptic functions sn, cn, dn (see [15], [16]).

We partition the domain C_2 into the two connected components

$$C_2^{\pm} = \{\lambda \in C \mid \alpha > 0,\ E \in (\alpha, +\infty),\ \operatorname{sgn} c = \pm 1\}.$$

Then

$$C_2^{\pm} = \left\{ k \in (0,1),\ \varphi\left(\operatorname{mod} \frac{2Kk}{\sqrt{\alpha}} \right),\ \alpha > 0,\ \beta(\operatorname{mod} 2\pi) \right\},$$

$$k = \sqrt{\frac{2\alpha}{E+\alpha}} = \left(\sqrt{\sin^2 \frac{\theta-\beta}{2} + \frac{c^2}{4\alpha}} \right)^{-1} \in (0,1),$$

$$\begin{cases} \sin \dfrac{\theta-\beta}{2} = s_2 \operatorname{sn} \dfrac{\sqrt{\alpha}\,\varphi}{k}; \\ \dfrac{c}{2} = s_2 \dfrac{\sqrt{\alpha}}{k} \operatorname{dn} \dfrac{\sqrt{\alpha}\,\varphi}{k}, \end{cases} \qquad s_2 = \operatorname{sgn} c.$$

In this case,

$$\cos \frac{\theta-\beta}{2} = \operatorname{cn} \frac{\sqrt{\alpha}\,\varphi}{k}.$$

In the domain C_2 we shall also use the coordinates (k, ψ, α, β), where $\psi = \varphi/k$. Then

$$C_2^{\pm} = \left\{ k \in (0,1),\ \psi\left(\operatorname{mod} \frac{2K}{\sqrt{\alpha}} \right),\ \alpha > 0,\ \beta(\operatorname{mod} 2\pi) \right\}.$$

We also partition the set C_3 into the two connected components

$$C_3^{\pm} = \{ \lambda \in C \mid \alpha > 0,\ E = \alpha,\ \operatorname{sgn} c = \pm 1 \}.$$

Then

$$C_3^{\pm} = \{ \varphi \in \mathbb{R},\ \alpha > 0,\ \beta(\operatorname{mod} 2\pi) \},$$

$$\begin{cases} \sin \dfrac{\theta-\beta}{2} = s_1 s_2 \tanh(\sqrt{\alpha}\,\varphi); \\ \dfrac{c}{2} = s_2 \dfrac{\sqrt{\alpha}}{\cosh(\sqrt{\alpha}\,\varphi)}, \end{cases} \qquad s_1 = \operatorname{sgn} \cos \frac{\theta-\beta}{2}, \quad s_2 = \operatorname{sgn} c.$$

In this case,

$$\cos \frac{\theta-\beta}{2} = s_1 \frac{1}{\cosh(\sqrt{\alpha}\,\varphi)}.$$

In Fig. 6 we depicted the grid of elliptic coordinates in the phase plane of the standard pendulum ($\alpha = 1$, $\beta = 0$). In the domain C_1 (oscillations of the pendulum with a low energy $E < 1$) we depicted the curves $k = \text{const}$, $\varphi/K = \text{const}$; in the domain C_2 (rotations of the pendulum with a higher energy $E > 1$) we depicted the curves $k = \text{const}$, $\psi/K = \text{const}$; these domains are separated by the set C_3 (motions of the pendulum with the critical energy $E = 1$) consisting of the two separatrices $k = 1$.

4.2. Elliptic coordinates in the inverse image of the exponential map. In accordance with the partition of the initial cylinder $C = \bigcup_{i=1}^{7} C_i$ we consider the partition of the inverse image of the exponential map

$$N = \bigcup_{i=1}^{7} N_i, \qquad N_i = C_i \times \mathbb{R}_+.$$

In similar fashion, along with the partition into the connected components

$$C_i = C_i^+ \cup C_i^-, \qquad C_i^\pm = C_i \cap \{\operatorname{sgn} c = \pm 1\}, \quad i = 2,\ 3,\ 6,$$

we shall consider the partitions

$$N_i = N_i^+ \cup N_i^-, \qquad N_i^\pm = C_i^\pm \times \mathbb{R}_+, \quad i = 2,\ 3,\ 6.$$

We define the elliptic coordinates in the subsets N_1, N_2, N_3:

$$\begin{aligned} N_1 &= \{(k, \varphi, \alpha, \beta, \delta)\}, \\ N_2 &= \{(k, \psi, \alpha, \beta, \delta)\}, \\ N_3 &= \{(\varphi, \alpha, \beta, \delta)\}, \\ \delta &= t\sqrt{\alpha}\,. \end{aligned}$$

Remark. The elliptic coordinates are suited to the action of continuous symmetries; rotations and dilatations have an especially simple form in these coordinates. Indeed, in the coordinates $N = \{(\theta, c, \alpha, \beta, t)\}$ we have

$$\vec{h}_0 = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \beta}\,, \qquad Z = -c\frac{\partial}{\partial c} - 2\alpha\frac{\partial}{\partial \alpha} + t\frac{\partial}{\partial t}\,;$$

hence in the coordinates $N_j = \{(k, \varphi, \alpha, \beta, \delta)\}$ we have

$$\vec{h}_0 = \frac{\partial}{\partial \beta}\,, \qquad Z = -2\alpha\frac{\partial}{\partial \alpha}\,.$$

The functions (k, φ, δ) are coordinates in the quotient spaces N_j'', $j = 1,\ 2,\ 3$.

4.3. Action of reflections in $\boldsymbol{N}$. We now describe the action of the reflections $\varepsilon^i \colon N_j \to N_j$ in the elliptic coordinates. Let $\nu \in N$; we set $\nu^i = \varepsilon^i(\nu) \in N$.

Proposition 4.1. 1) *If* $\nu = (k, \varphi, \alpha, \beta, t) \in N_1$, *then* $\nu^i = (k, \varphi^i, \alpha, \beta^i, t) \in N_1$, $i = 1,\ 2,\ 3$, *and*

$$\begin{aligned} \varphi^1 + \varphi_t &= \frac{2K}{\sqrt{\alpha}} \left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right), & \beta^1 &= \beta, \\ \varphi^2 + \varphi_t &= 0 \left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right), & \beta^2 &= -\beta, \\ \varphi^3 - \varphi &= \frac{2K}{\sqrt{\alpha}} \left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right), & \beta^3 &= -\beta. \end{aligned}$$

2) *If* $\nu = (k, \psi, \alpha, \beta, t) \in N_2$, *then* $\nu^i = (k, \psi^i, \alpha, \beta^i, t) \in N_2$, $i = 1, 2, 3$; *moreover*

$$\nu \in N_2^\pm \;\Rightarrow\; \nu^1 \in N_2^\mp,\ \nu^2 \in N_2^\pm,\ \nu^3 \in N_2^\mp,$$

and

$$\begin{aligned} \psi^1 + \psi_t &= 0 \left(\operatorname{mod} \frac{2K}{\sqrt{\alpha}}\right), & \beta^1 &= \beta, \\ \psi^2 + \psi_t &= 0 \left(\operatorname{mod} \frac{2K}{\sqrt{\alpha}}\right), & \beta^2 &= -\beta, \\ \psi^3 + \psi &= 0 \left(\operatorname{mod} \frac{2K}{\sqrt{\alpha}}\right), & \beta^3 &= -\beta. \end{aligned}$$

3) *If* $\nu = (\varphi, \alpha, \beta, t) \in N_3$, *then* $\nu^i = (\varphi^i, \alpha, \beta^i, t) \in N_3$, $i = 1, 2, 3$; *moreover,*

$$\nu \in N_3^{\pm} \;\Rightarrow\; \nu^1 \in N_3^{\mp}, \quad \nu^2 \in N_3^{\pm}, \quad \nu^3 \in N_3^{\mp},$$

and

$$\begin{aligned} \varphi^1 + \varphi_t &= 0, & \beta^1 &= \beta, \\ \varphi^2 + \varphi_t &= 0, & \beta^2 &= -\beta, \\ \varphi^3 + \varphi &= 0, & \beta^3 &= -\beta. \end{aligned}$$

Proposition 4.1 is illustrated in Fig. 7–9 (in the case $\alpha = 1$, $\beta = 0$).

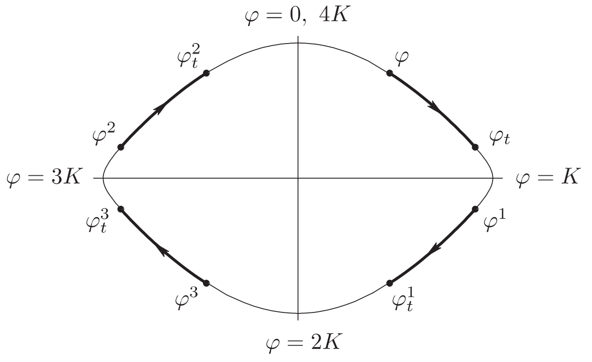


Figure 7. Reflections in C_1

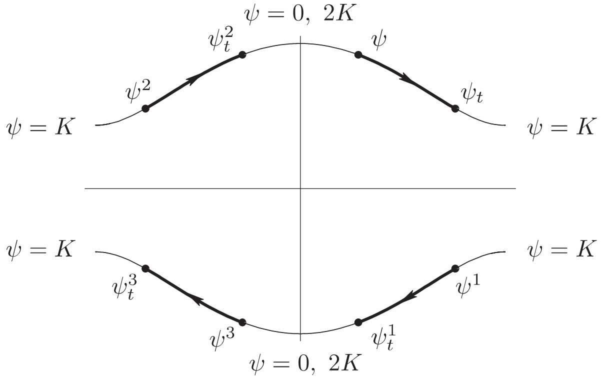


Figure 8. Reflections in C_2

Proof. We prove only part 1), since the other two parts can be proved in similar fashion.

The reflections ε^i preserve the domain N_1 due to the fact that

$$\varepsilon^i \colon E \to E, \qquad \varepsilon^1, \varepsilon^3 \colon c \mapsto -c, \qquad \varepsilon^2 \colon c \mapsto c,$$

which follows from equalities (5)–(7).

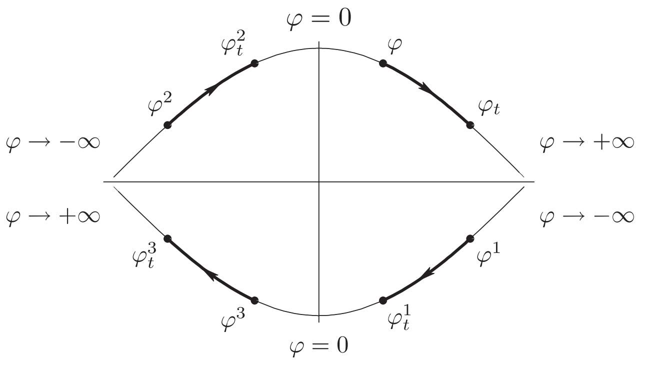


Figure 9. Reflections in C_3

Next, from equality (5) we obtain $\theta^1 = \theta_t$ and $c^1 = -c_t$, whence, taking into account the construction of elliptic coordinates (§ 4.1) we have

$$\operatorname{sn}(\sqrt{\alpha}\,\varphi^1) = \operatorname{sn}(\sqrt{\alpha}\,\varphi_t), \qquad \operatorname{cn}(\sqrt{\alpha}\,\varphi^1) = -\operatorname{cn}(\sqrt{\alpha}\,\varphi_t);$$

hence,

$$\varphi_1 + \varphi_t = \frac{2K}{\sqrt{\alpha}} \left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right).$$

The expressions for the action of the other reflections in the elliptic coordinates can be obtained in similar fashion.

§ 5. The action of reflections in the image of the exponential map

5.1. Decomposition and coordinates in M and M''. In order to obtain a simple description of the action of reflections in the image of the exponential map it is convenient to use special coordinates in M suited to the action of the symmetries X_0 and Y. We now describe their construction.

Recall that

$$M = \{q\} = \mathbb{R}^5_{x,y,z,v,w}, \qquad q_0 = (0,0,0,0,0).$$

We introduce polar coordinates in the planes (x, y) and (v, w):

$$x = r\cos\chi, \quad y = r\sin\chi; \qquad v = \rho\cos\omega, \quad w = \rho\sin\omega.$$

In the domain $\{r > 0,\ \rho > 0\}$ the angle

$$\gamma = \chi - \omega$$

is defined. We consider the following subsets of M:

$$M = M_0 \cup \{q_0\}, \qquad M_0 = M \setminus \{q_0\} = \{r^2 + z^2 + \rho^2 > 0\},$$
$$M_0 = M_1 \cup M_2 \cup M_3^+ \cup M_3^-,$$
$$M_1 = \{r > 0\}, \quad M_2 = \{\rho > 0\}, \quad M_3^\pm = \{\pm z > 0\}.$$

We describe the desired coordinates in the charts M_1, M_2, $M_3^\pm$ of the manifold M_0:

$$M_1 = \{(r > 0,\ \chi \in S^1,\ P, Q, R)\},$$
$$P = \frac{z}{2r^2}, \quad Q = \frac{xv + yw}{r^4} = \frac{\rho}{r^3}\cos\gamma, \quad R = \frac{-yv + xw}{r^4} = \frac{\rho}{r^3}\sin\gamma;$$
$$M_2 = \{(\rho > 0,\ \omega \in S^1,\ P', Q', R')\},$$
$$P' = \frac{z}{2\rho^{2/3}}, \quad Q' = \frac{xv + yw}{\rho^{4/3}} = \frac{r}{\rho^{1/3}}\cos\gamma, \quad R' = \frac{-yv + xw}{\rho^{4/3}} = \frac{r}{\rho^{1/3}}\sin\gamma;$$
$$M_3^\pm = \{(r' \geqslant 0,\ \chi \in S^1,\ \rho' \geqslant 0,\ \omega \in S^1)\},$$
$$r' = \frac{r}{|z|^{1/2}}, \qquad \rho' = \frac{\rho}{|z|^{3/2}}.$$

In addition,

$$\begin{aligned} M_3^\pm &= \{(\pm z > 0,\ \chi \in S^1,\ r' \geqslant 0,\ \rho' \geqslant 0,\ \gamma \in S^1)\} \\ &= \{(\pm z > 0,\ \chi \in S^1,\ P'', Q'', R'')\}, \end{aligned}$$
$$P'' = \frac{r'^2 - \rho'^2}{\sqrt{r'^2 + \rho'^2}}, \quad Q'' = \frac{r'\rho'}{\sqrt{r'^2 + \rho'^2}}\cos\gamma, \quad R'' = \frac{r'\rho'}{\sqrt{r'^2 + \rho'^2}}\sin\gamma.$$

In the coordinates $(r, \chi, z, \rho, \omega)$ in the domain $M_1 \cap M_2 = \{r > 0, \rho > 0\}$ the continuous symmetries take the form

$$X_0 = \frac{\partial}{\partial\chi} + \frac{\partial}{\partial\omega}, \qquad Y = r\frac{\partial}{\partial r} + 2z\frac{\partial}{\partial z} + 3\rho\frac{\partial}{\partial\rho}.$$

In the coordinates (r, χ, P, Q, R) in the domain M_1 we obtain

$$X_0 = \frac{\partial}{\partial\chi}, \qquad Y = r\frac{\partial}{\partial r};$$

hence the quotient space M_1'' is parametrized by the coordinates (P, Q, R). Similarly, in the domain $M_2 = \{(\rho, \omega, P', Q', R')\}$ we have

$$X_0 = \frac{\partial}{\partial\omega}, \qquad Y = 3\rho\frac{\partial}{\partial\rho}$$

and therefore $M_2'' = \{(P', Q', R')\}$, while in the domains $M_3^\pm = \{z, \chi, P'', Q'', R'')\}$ we have

$$X_0 = \frac{\partial}{\partial\chi}, \qquad Y = 2z\frac{\partial}{\partial z}$$

and therefore $M_3^{\pm\prime\prime} = \{(P'', Q'', R'')\}$.

We consider the following partition of M into invariant sets of the group of continuous and discrete symmetries G:

$$M = \{q_0\} \cup M_1 \cup M_4 \cup M_5^{\pm},$$

where the domain $M_1 = M \setminus \{q_0\} = \{r > 0\}$ was defined above, while

$$M_4 = M_2 \setminus M_1 = \{r = 0,\ \rho > 0\},$$
$$M_5^{\pm} = M_3^{\pm} \setminus (M_1 \cup M_4) = \{r = 0,\ \rho = 0,\ \pm z > 0\}.$$

The coordinates in M_1 and M_1'' were described above, while

$$M_4 = \{(\rho > 0,\ \omega \in S^1,\ P')\}, \qquad M_4'' = \{P'\},$$
$$M_5^{\pm} = \{\pm z > 0\}, \qquad M_5^{\pm\,''} = \{z = \pm 1\}.$$

5.2. Action of reflections in M. We describe the action of the reflections

$$\varepsilon^i \colon M \to M, \qquad q = (x, y, z, v, w) \mapsto q^i = (x^i, y^i, z^i, v^i, w^i)$$

in the coordinates introduced in § 5.1.

Proposition 5.1. *We have*

1) $q = q_0 \ \Rightarrow\ q^i = q_0,\ i = 1,\ 2,\ 3;$
2) $q = (r, \chi, P, Q, R) \in M_1 \ \Rightarrow\ q^i = (r, \chi^i, P^i, Q^i, R) \in M_1,$ *and*

$$\chi^1 = \chi\ (\mathrm{mod} 2\pi), \qquad P^1 = -P, \qquad Q^1 = Q - 2P,$$
$$\chi^2 + \chi = 0\ (\mathrm{mod} 2\pi), \qquad P^2 = P, \qquad Q^2 = -Q + 2P,$$
$$\chi^3 + \chi = 0\ (\mathrm{mod} 2\pi), \qquad P^3 = -P, \qquad Q^3 = Q - 2P;$$

3) $q = (\rho, \omega, P') \in M_4 \ \Rightarrow\ q^i = (\rho, \omega^i, P'^i) \in M_4,$ *and*

$$\omega^1 = \omega\ (\mathrm{mod} 2\pi), \qquad P'^1 = -P',$$
$$\omega^2 + \omega = \pi\ (\mathrm{mod} 2\pi), \qquad P'^2 = P',$$
$$\omega^3 + \omega = \pi\ (\mathrm{mod} 2\pi), \qquad P'^3 = -P';$$

4) $q = z \in M_5 \ \Rightarrow\ q^i = z^i \in M_5,$ *and*

$$q \in M_5^{\pm} \quad\Rightarrow\quad q^1 \in M_5^{\mp},\ q^2 \in M_5^{\pm},\ q^3 \in M_5^{\mp},$$
$$z^1 = -z, \quad z^2 = z, \quad z^3 = -z.$$

Proof. By Proposition 2.5 the reflections act in M as follows:

$$\varepsilon^1 \colon (x, y, z, v, w) \mapsto (x, y, -z, v - xz, w - yz),$$
$$\varepsilon^2 \colon (x, y, z, v, w) \mapsto (x, -y, z, -v + xz, w - yz),$$
$$\varepsilon^3 \colon (x, y, z, v, w) \mapsto (x, -y, -z, -v, w).$$

The assertion of part 1) is obvious; we now prove part 2). It is immediately clear that the reflections preserve the distance r, and therefore also the domain M_1.

Next, the reflection ε^1 does not alter the vector (x,y), and therefore it does not alter the angle χ either. Finally,

$$P^1 = \frac{-z}{2r^2} = -P,$$
$$Q^1 = \frac{x(v-xz)+y(w-yz)}{r^4} = \frac{xv+yw}{r^4} - \frac{(x^2+y^2)z}{r^4} = Q-2P,$$
$$R^1 = \frac{-y(v-xz)+x(w-yz)}{r^4} = \frac{-yv+xw}{r^4} = R.$$

The remaining assertions can be proved in similar fashion.

Remark. It is clear from part 2) of Proposition 5.1 that in the domain M_1 (which is an open everywhere dense subset of M) the reflections ε^i can be particularly simply represented in the following coordinates in the quotient space M_1'':

$$P = \frac{z}{2r^2}\,, \qquad Q-P = \frac{xv+yw-zr^2/2}{r^4}\,, \qquad R = \frac{-yv+xw}{r^4}\,;$$

namely,

$$\begin{aligned}
\varepsilon^1 &: \quad P\mapsto -P, \quad Q-P\mapsto Q-P, \quad R\mapsto R,\\
\varepsilon^2 &: \quad P\mapsto P, \quad Q-P\mapsto -Q+P, \quad R\mapsto R,\\
\varepsilon^3 &: \quad P\mapsto -P, \quad Q-P\mapsto Q-P, \quad R\mapsto R.
\end{aligned}$$

In other words, in the plane $(P, Q-P)$ the discrete symmetries ε^i act as the reflections in the coordinate axes $P=0$ and $Q-P=0$. This explains the important role that the surfaces $z=0$ and $V=xv+yw-zr^2/2=0$ will play in the study of the Maxwell strata. The subsequent papers [10], [11] are devoted to this study.

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