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The Maxwell set in the generalized Dido problem

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Abstract. The generalized Dido problem is considered — a model of the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$. We study the Maxwell set, that is, the locus of the intersection points of geodesics of equal lengths. A general description is obtained for the Maxwell strata corresponding to the symmetry group of the exponential map generated by rotations and reflections. The invariant and graphic meaning of these strata is clarified.

Bibliography: 19 titles.

§ 1. Introduction

1.1. Statement of the problem. The present paper is devoted to the study of optimality of geodesics in the generalized Dido problem. The problem can be formulated as follows. Suppose that we are given two points $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ connected by some curve $\gamma_0 \subset \mathbb{R}^2$, a number $S \in \mathbb{R}$, and a point $c = (c_x, c_y) \in \mathbb{R}^2$. One needs to find a shortest curve $\gamma \subset \mathbb{R}^2$ connecting the points (x_0, y_0) and (x_1, y_1) such that the domain bounded by the two curves γ_0 and γ has the prescribed algebraic area S and centre of mass c.

It was shown in [1] that this problem can be reformulated as an optimal control problem in 5-dimensional space with 2-dimensional control and the integral criterion

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q = (x, y, z, v, w) \in M = \mathbb{R}^5, \quad u = (u_1, u_2) \in U = \mathbb{R}^2,$$
$$q(0) = q_0 = 0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$

where the vector fields for the controls have the form

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2 + y^2}{2}\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2 + y^2}{2}\frac{\partial}{\partial v}.$$

From the invariant viewpoint, this is the sub-Riemannian problem defined by the distribution

$$\Delta_q = \operatorname{span}(X_1(q), X_2(q)), \qquad q \in M,$$

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with a scalar product $\langle \cdot , \cdot \rangle$ in which the fields X_1, X_2 form an orthonormal basis:

$$\langle X_i, X_j \rangle = \delta_{ij}, \qquad i, j = 1, 2.$$

The Lie algebra generated by the fields X_1, X_2 is a free nilpotent Lie algebra of length 3 with two generators. The distribution Δ has the flag

$$\Delta \subset \Delta^2 = [\Delta, \Delta] \subset \Delta^3 = [\Delta, \Delta^2] = TM$$

and the growth vector $(2, 3, 5) = (\dim \Delta_q, \dim \Delta_q^2, \dim \Delta_q^3)$.

Thus, $(\Delta, \langle \cdot , \cdot \rangle)$ is a nilpotent sub-Riemannian structure with the growth vector $(2, 3, 5)$. This structure is a local quasihomogeneous nilpotent approximation of an arbitrary sub-Riemannian structure on a 5-dimensional manifold with the growth vector $(2, 3, 5)$ (see [2], [3], as well as [4]). As shown in [5], such a nilpotent structure is unique. The generalized Dido problem is a model of the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$.

1.2. Known results. We continue the study of the generalized Dido problem started in [1], [5], [6].

In [5] and [6], respectively, the groups of continuous and discrete symmetries in this problem were found: there is a two-parameter continuous symmetry group (rotations and dilatations), as well as a discrete symmetry group of order 4 (reflections).

A parametrization of the sub-Riemannian geodesics (extremal trajectories) by the Jacobi elliptic functions was obtained in [1]. The abnormal geodesics are optimal up to infinity, while the normal ones, generally speaking, are optimal on finite time intervals.

1.3. Contents of the paper. A point at which a geodesic ceases to be optimal is called a *cut point*. It is known that a normal geodesic can cease to be optimal either because another geodesic with the same value of the functional comes to some point of it (a Maxwell point), or because a family of geodesics has an envelope (a conjugate point). In the present paper we find the Maxwell points corresponding to the symmetry group preserving time on the geodesics (rotations and reflections). Namely, we find two hypersurfaces in the state space M containing all such Maxwell points. Computer-aided calculations show that it is on these hypersurfaces that geodesics cease to be optimal. We clarify the invariant meaning of these hypersurfaces in terms of the sub-Riemannian structure, as well as their graphic meaning for the Euler elastics (the projections of geodesics onto the plane (x, y)).

Localization of the intersection points of geodesics with the hypersurfaces found in this paper will be the subject of the subsequent paper [7]. For that we shall need the more complicated technique of elliptic functions, which is avoided in this paper.

We used the system "Mathematica" [8] to carry out complicated calculations and to produce the illustrations in this paper.

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§ 2. The Maxwell set

We begin by recalling some notions and notation introduced in the preceding papers [1], [6]. It follows from the Pontryagin maximum principle [9] that the extremals in the generalized Dido problem are the trajectories of the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, $\lambda \in T^*M$, with the Hamiltonian $H = (h_1^2 + h_2^2)/2$, where $h_i(\lambda) = \langle \lambda, X_i(q) \rangle$. The geodesics are the projections of the extremals in the cotangent bundle T^*M onto the state space M: $q_t = \pi(\lambda_t)$, $\lambda_t = e^{t\vec{H}}(\lambda)$. Henceforth, $e^{t\vec{H}}$ denotes the flow of the Hamiltonian field $\vec{H}$ with the Hamiltonian H. Since the Hamiltonian H is homogeneous, it is sufficient to consider the restriction of the Hamiltonian flow to the level surface $H = 1/2$ and therefore it is sufficient to take initial covectors λ in the initial cylinder $C = \{H = 1/2\} \cap T^*_{q_0}M$. All the information about the geodesics is contained in the exponential map $\operatorname{Exp}\colon C \times \mathbb{R}_+ \to M$ given by $\operatorname{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda) = q_t$.

2.1. Optimality of normal geodesics. The *Maxwell set* in the inverse image of the exponential map is defined as follows:

$$\mathrm{MAX} = \bigl\{(\lambda, t) \in C \times \mathbb{R}_+ \mid \exists\, \widetilde{\lambda} \in C,\ \widetilde{\lambda} \neq \lambda \colon \operatorname{Exp}(\widetilde{\lambda}, t) = \operatorname{Exp}(\lambda, t)\bigr\}.$$

An inclusion $(\lambda, t_1) \in \mathrm{MAX}$ means that two distinct geodesics

$$q_s = \operatorname{Exp}(\lambda, s) \not\equiv \widetilde{q}_s = \operatorname{Exp}(\widetilde{\lambda}, s), \qquad s \in [0, t_1],$$

of the same sub-Riemannian length $l = t_1$ intersect at the point $q_{t_1} = \widetilde{q}_{t_1}$ (see Fig. 1).

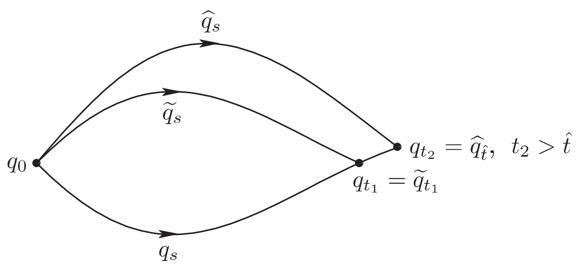

Figure 1. Non-optimalality of the geodesic q_s after the Maxwell point q_{t_1}

The Maxwell set is closely related to optimality of geodesics: a geodesic cannot be optimal after an intersection with another geodesic of the same length. The following proposition was proved by Jacquet [10]. We give it with proof for the sake of completeness of the exposition.

Proposition 2.1. *Let q_s and $\widetilde{q}_s$ be two distinct geodesics: $q_s \not\equiv \widetilde{q}_s$, $s \in [0, t_1]$. If $q_{t_1} = \widetilde{q}_{t_1}$, then for any $t_2 > t_1$ the geodesic q_s, $s \in [0, t_2]$, is not optimal.*

In other words, for any $t_2 > t_1$ there exists a geodesic $\widehat{q}_s$, $s \in [0, \hat{t}\,]$, of a smaller sub-Riemannian length than that of q_s, $s \in [0, t_2]$, connecting q_0 and q_{t_2} (see Fig. 1). Therefore any geodesic $q_s = \operatorname{Exp}(\lambda, s)$ is non-optimal after the Maxwell time t_1, $(\lambda, t_1) \in \mathrm{MAX}$.

Proof. We prove Proposition 2.1 arguing by contradiction. Suppose that for some $t_2 > t_1$ the geodesic q_s, $s \in [0, t_2]$, is optimal. Then the broken geodesic

$$q'_s = \begin{cases} \widetilde{q}_s, & s \in [0, t_1]; \\ q_s, & s \in [t_1, t_2] \end{cases}$$

is also optimal. But all the geodesics in the generalized Dido problem are analytic curves. Therefore it follows from the identity $q'_s \equiv q_s$, $s \in [t_1, t_2]$, that $q'_s \equiv q_s$, $s \in [0, t_2]$, whence $\widetilde{q}_s \equiv q_s$, $s \in [0, t_1]$; a contradiction.

Remark. The exponential map is a Lagrangian map [11]. It follows from the theory of Lagrangian singularities that in our problem normal geodesics cease to be optimal at the *first* Maxwell point (that is, they are optimal up to and including this point and non-optimal after it). However, it is difficult to find the first Maxwell point. We shall investigate optimality of normal geodesics as follows:

1) we shall find certain subsets of the Maxwell set — the Maxwell strata MAX_i, $i = 0, \dots, 3$, generated by the symmetries of the exponential map that do not alter time (the rotation $\vec{h}_0$ and the reflections ε^i);
2) we shall prove that along every normal extremal the first conjugate point is encountered not earlier than the first intersection with the Maxwell strata MAX_i;
3) we shall prove that the first point on a normal extremal in the Maxwell strata MAX_i is a cut point (a point of loss of optimality).

This method of investigation of optimality was successfully applied for solving several problems of sub-Riemannian geometry [12], [13]. In the present paper we solve problem 1).

As noted in the book [11], the term *Maxwell set* originates "in connection with the Maxwell rule of the van der Waals theory, according to which phase transition takes place at a value of the parameter for which two maxima of a certain smooth function are equal to each other".

2.2. The Maxwell strata generated by rotations and reflections. In [6] we defined and studied the reflections ε^i, $i = 1, 2, 3$, — the discrete symmetries of the exponential map $\varepsilon^i\colon N \to N$, $\varepsilon^i\colon M \to M$, $\mathrm{Exp} \circ \varepsilon^i = \varepsilon^i \circ \mathrm{Exp}$. We set $\nu^i = \varepsilon^i(\nu)$ for $\nu = (\lambda, t) \in N = C \times \mathbb{R}_+$. Along with the discrete symmetry group $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$, the exponential map has the continuous two-parameter symmetry group $G_{\vec{h}_0, Z} = e^{\mathbb{R}\vec{h}_0} \circ e^{\mathbb{R}Z}$ (see [1]), where

$$h_0(\lambda) = \langle \lambda, X_0(q) \rangle, \qquad X_0 = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} - w\,\frac{\partial}{\partial v} + v\,\frac{\partial}{\partial w}\,,$$

$$Z = \vec{h}_Y + e, \qquad h_Y(\lambda) = \langle \lambda, Y(q) \rangle,$$

$$Y = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + 2z\,\frac{\partial}{\partial z} + 3v\,\frac{\partial}{\partial v} + 3w\,\frac{\partial}{\partial w}\,, \qquad e = \sum_{i=1}^{5} h_i\,\frac{\partial}{\partial h_i}\,.$$

We define the *Maxwell strata generated by the rotations* $\vec{h}_0$ *and the reflections* ε^i:

$$\mathrm{MAX}_0 = \bigl\{\nu \in N \mid \exists\, \sigma \in \mathbb{R}\colon\ \widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu) \neq \nu,\ \mathrm{Exp}(\widetilde{\nu}) = \mathrm{Exp}(\nu)\bigr\},$$
$$\mathrm{MAX}_i = \bigl\{\nu \in N \mid \exists\, \sigma \in \mathbb{R}\colon\ \widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu^i) \neq \nu,\ \mathrm{Exp}(\widetilde{\nu}) = \mathrm{Exp}(\nu)\bigr\}, \quad i = 1, 2, 3.$$

By the definition of the Maxwell set,

$$\mathrm{MAX}_i \subset \mathrm{MAX}, \qquad i = 0, 1, 2, 3.$$

Here we give a general description of the sets MAX_i, while the detailed description of these sets will be obtained in the subsequent paper [7].

From a more general viewpoint we shall analyse the Maxwell set corresponding to the symmetry group generated by rotations and reflections. Let G be some group acting in the inverse image and image of the exponential map:

$$g\colon N \to N, \quad g\colon M \to M, \qquad g \in G.$$

The group G is called a *symmetry group of the exponential map* if for any $g \in G$ the following diagram is commutative:

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "\mathrm{Exp}"'] & N \arrow[d, "\mathrm{Exp}"] \\ M \arrow[r, "g"] & M, \end{tikzcd}$$

that is, $\mathrm{Exp} \circ g = g \circ \mathrm{Exp}$. Suppose that the group G preserves time:

$$g(\lambda, t) = (\lambda', t), \qquad (\lambda, t) \in N, \quad g \in G.$$

We define the *Maxwell set corresponding to the group* G to be the set

$$\mathrm{MAX}_G = \bigl\{\nu \in N \mid \exists\, g \in G\colon\ \widetilde{\nu} = g(\nu) \neq \nu,\ \mathrm{Exp}(\widetilde{\nu}) = \mathrm{Exp}(\nu)\bigr\}.$$

Clearly, $\mathrm{MAX}_G \subset \mathrm{MAX}$ for any symmetry group of the exponential map that preserves time. (One can define the Maxwell set for a group acting only in the inverse image of the exponential map; but if the action of the group is undefined in the image of the exponential map, then finding this set seems to be difficult.)

In our problem, the group generated by rotations and reflections

$$G = G_{\vec{h}_0,\varepsilon} = \langle e^{s\vec{h}_0}, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$$

is a symmetry group of the exponential map preserving time. By the commutation rules in the group $G_{\vec{h}_0,\varepsilon}$ ([6], Proposition 3.2) every element of this group has the form $g = e^{s\vec{h}_0}$ or $g = e^{s\vec{h}_0} \circ \varepsilon^i$; hence,

$$\mathrm{MAX}_{G_{\vec{h}_0,\varepsilon}} = \bigcup_{i=0}^{3} \mathrm{MAX}_i\,.$$

2.3. Factorization of the Maxwell strata. The strata MAX_i are factorized by the action of the group $G_{\vec{h}_0,Z} = e^{\mathbb{R}\vec{h}_0} \circ e^{\mathbb{R}Z}$ in the same fashion as the whole Maxwell set:

$$\begin{array}{ccccc} \mathrm{MAX}_i & \xrightarrow{\mathrm{Id}} & N & \xrightarrow{\mathrm{Exp}} & M \\ \big\downarrow{\scriptstyle \pi_0''} & & \big\downarrow{\scriptstyle \pi_0''} & & \big\downarrow{\scriptstyle \pi_1''} \\ \mathrm{MAX}_i'' & \xrightarrow{\mathrm{Id}} & N'' & \xrightarrow{\mathrm{Exp}'} & M''. \end{array}$$

This is a consequence of the following assertion.

Proposition 2.2. *The Maxwell strata* MAX_i *are invariant under rotations and dilatations*:

$$e^{s\vec{h}_0} \circ e^{rZ}(\mathrm{MAX}_i) = \mathrm{MAX}_i, \qquad i = 0, \dots, 3, \quad s, r, \in \mathbb{R}.$$

Proof. In the case $i = 0$ the assertion obviously follows from the rules of commutation of the continuous symmetries and the exponential map ([6], Proposition 3.1).

We consider the case $i = 2$ (as Proposition 3.1 in [6] shows, the cases $i = 1, 3$ are simpler than this one). Let $\nu \in \mathrm{MAX}_2$; then for some $\sigma \in \mathbb{R}$ the following conditions hold:

$$\widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu^2) \neq \nu, \qquad \mathrm{Exp}(\widetilde{\nu}) = \mathrm{Exp}(\nu).$$

It is easy to see that then for $\nu_1 = e^{s\vec{h}_0} \circ e^{rZ}(\nu)$ and $\sigma_1 = \sigma + 2s$ the conditions

$$\widetilde{\nu}_1 = e^{\sigma_1\vec{h}_0}(\nu_1^2) \neq \nu_1, \qquad \mathrm{Exp}(\widetilde{\nu}_1) = \mathrm{Exp}(\nu_1)$$

hold, that is, $\nu_1 = e^{s\vec{h}_0} \circ e^{rZ}(\nu) \in \mathrm{MAX}_2$.

§ 3. Multiple points of the exponential map

Each Maxwell stratum MAX_i consists of multiple points of the exponential map that are not fixed points for the corresponding continuous or discrete symmetry. In this and the following sections we find, respectively, the multiple points of the exponential map and the fixed points of symmetries.

We define the following function on the state space, which is important for what follows:

$$V = xv + yw - z\,\frac{r^2}{2}\,.$$

The origin of this function was explained in [6] (see the remark at the end of § 5.2). This function is invariant under rotations and is homogeneous of order 4 under dilatations:

$$X_0 V = 0, \qquad YV = 4V. \tag{1}$$

It follows from Proposition 2.5 in [6] that the function V is preserved, up to a sign, by reflections:

$$V \circ \varepsilon^1 = V, \qquad V \circ \varepsilon^2 = -V, \qquad V \circ \varepsilon^3 = -V.$$

Note that similar properties are also enjoyed by the function z:

$$X_0 z = 0, \qquad Yz = 2z, \tag{2}$$

$$z \circ \varepsilon^1 = -z, \qquad z \circ \varepsilon^2 = z, \qquad z \circ \varepsilon^3 = -z.$$

3.1. Derivation of equations for multiple points. Recall [6] that we use the notation $q^i = \varepsilon^i(q)$, where $q, q^i \in M$. In the planes (x, y) and (v, w) we use the polar coordinates: $x = r\cos\chi$, $y = r\sin\chi$, $v = \rho\cos\omega$, $w = \rho\sin\omega$.

In the following assertion we obtain equations of the hypersurfaces containing the Maxwell strata MAX_i.

Proposition 3.1. *The following hold*:

$$1)\quad e^{\sigma X_0}(q^1) = q \qquad\Leftrightarrow\qquad z = 0, \qquad \begin{cases} \sigma = 0 & \textit{for } r^2+\rho^2 > 0; \\ \forall\,\sigma & \textit{for } r^2+\rho^2 = 0, \end{cases}$$

$$2)\quad e^{\sigma X_0}(q^2) = q \qquad\Leftrightarrow\qquad V = 0, \qquad \begin{cases} \sigma = 2\chi & \textit{for } r > 0; \\ \sigma = 2\omega-\pi & \textit{for } r = 0,\ \rho > 0; \\ \forall\,\sigma & \textit{for } r = \rho = 0, \end{cases}$$

$$3)\quad e^{\sigma X_0}(q^3) = q \qquad\Leftrightarrow\qquad z = V = 0, \qquad \begin{cases} \sigma = 2\chi & \textit{for } r > 0; \\ \sigma = 2\omega-\pi & \textit{for } r = 0,\ \rho > 0; \\ \forall\,\sigma & \textit{for } r = \rho = 0. \end{cases}$$

Proof. We use the formulae for the action of reflections in coordinates on M (see [6], Proposition 2.5).

1) The assertion follows from the relations

$$\begin{aligned} \varepsilon^1 \colon (x,y,z,v,w) &\mapsto (x,\ y,\ -z,\ v-xz,\ w-yz), \\ e^{sX_0} \colon (x,y,z,v,w) &\mapsto (x\cos s - y\sin s,\ x\sin s + y\cos s,\ z, \\ &\qquad\qquad v\cos s - w\sin s,\ v\sin s + w\cos s). \end{aligned} \tag{3}$$

2) Recall that

$$\varepsilon^2 \colon (x,y,z,v,w) \mapsto (x,-y,-z,-v+xz,w-yz).$$

Let $r > 0$. Then the equality $(x,y) = (x\cos s + y\sin s,\ x\sin s - y\cos s)$ means that $s = 2\chi$, that is,

$$\cos s = \frac{x^2-y^2}{x^2+y^2}, \qquad \sin s = \frac{2xy}{x^2+y^2}.$$

But then the system of equations

$$(-v+xz)\cos s - (w-yz)\sin s = v, \qquad (-v+xz)\sin s + (w-yz)\cos s = w$$

can be rewritten in the form

$$xV = 0, \qquad yV = 0,$$

which is equivalent to the equality $V = 0$.

Let $r = 0$, $\rho > 0$. In this case the equality

$$(v,w) = (-v\cos s + w\sin s,\ -v\sin s + w\cos s)$$

means that $s = 2\omega - \pi$. Moreover, the equality $r = 0$ implies $V = 0$.

Finally, in the case $r = \rho = 0$ both equalities hold: $e^{sX_0}(q^2) = q$ for any σ, as well as $V = 0$.

3) Recall that

$$\varepsilon^3 \colon (x, y, z, v, w) \mapsto (x, -y, -z, -v, w).$$

The assertion follows from part 2) by the fact that the reflections ε^3 and ε^2 coincide under the condition $z = 0$, which is necessary for the equality $e^{sX_0}(q^3) = q$ to hold.

To describe the Maxwell stratum MAX_0 we shall need the following assertion, which obviously follows from formula (3).

Proposition 3.2. *Let $s \neq 2\pi n$, $n \in \mathbb{N}$. Then*

$$e^{\sigma X_0}(q) = q \quad \Leftrightarrow \quad r^2 + \rho^2 = 0.$$

3.2. The graphic meaning of multiple points for the Euler elastics. As shown in [1], in the generalized Dido problem the projections of normal geodesics q_s onto the plane (x, y) satisfy the differential equations

$$\dot{x}_s = \cos\theta_s,$$
$$\dot{y}_s = \sin\theta_s,$$

where the angle θ_s in turn is a solution of the pendulum equation

$$\ddot{\theta}_s = -\alpha\sin(\theta_s - \beta), \qquad \alpha, \beta = \mathrm{const}. \tag{4}$$

Such curves (x_s, y_s) are called *Euler elastics*; they were discovered by Euler as the stationary profiles of an elastic rod. The elastics are extremals of the functional $\dfrac{1}{2}\displaystyle\int_\gamma \varkappa^2(s)\,ds$ for planar curves, where $\varkappa$ is the curvature of a curve γ (see, for example, [14], [15]). The Euler elastics (the solutions of the generalized Dido problem) form, up to rotations and dilatations, a one-parameter family of curves connecting the straight line and the circle (the solutions of the classical Dido problem [16]). Depending on the value of the energy $E = c^2/2 - \alpha\cos(\theta - \beta)$ of the pendulum (4), elastics can be inflectional (with inflection points) for $E < \alpha$ (pendulum's oscillations with low energy), and non-inflectional for $E > \alpha$ (pendulum's rotations with high energy). Corresponding to the critical value of the energy $E = \alpha$ there is a critical non-inflectional elastic (for $\theta - \beta \neq \pi$, a non-periodic motion of the pendulum on the separatrix) and a straight line (for $\theta - \beta = \pi$, the unstable equilibrium position of the pendulum. Finally, corresponding to the minimum of the energy $E = -\alpha$ there is also an elastic that is a straight line (the stable equilibrium position of the pendulum). Sketches of elastics of various types are given in [1].

The functions z, V and the equations $z = 0$, $V = 0$, as well as the equation $r^2 + \rho^2 = 0$ have a graphic meaning for the Euler elastics.

We complete the arc of an elastic with the initial point O and end-point R to a closed curve by constructing the segment RO (see Fig. 2). Then the function z at the point R is equal to the algebraic area of the oriented domain bounded by the arc of the elastic OR and its chord RO, that is, to the algebraic sum of the areas of the connected components of this domain (the area of a component is taken with

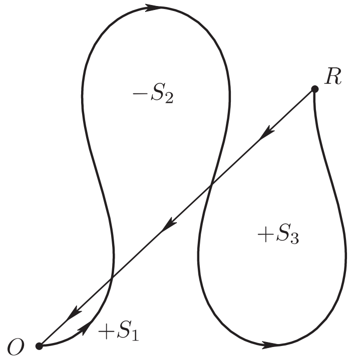

Figure 2. $z = +S_1 - S_2 + S_3 - \cdots$

sign "+" if we go round it in the positive direction, and with sign "−" if we go round it in the negative direction). Indeed, along the elastic we have

$$z = \frac{1}{2}\int x\,dy - y\,dx = \frac{1}{2}\int r^2\,d\chi = \sum \pm S_i.$$

The equality $z = 0$ means that the arc of the elastic and its chord bound a domain of zero algebraic area. This equality trivially holds for the arcs of an elastic centred at its inflection points (see § 4.2, Fig. 5). Such elastics are fixed points of the reflection $\varepsilon^1 \colon N_1 \mapsto N_1$. For finding the Maxwell points it is important to find the non-trivial solutions of the equation $z = 0$; it is such an arc of an elastic that is depicted in Fig. 2: for this arc, $z = S_1 - S_2 + S_3 = 0$.

We now clarify the graphic meaning of the variable V. Using the formulae for the coordinates of the centre of mass of a segment of an elastic for $z \neq 0$ (see [1])

$$c_x = \frac{1}{z}\biggl(v - \frac{r^2}{6}\,y\biggr), \qquad c_y = \frac{1}{z}\biggl(w + \frac{r^2}{6}\,x\biggr), \tag{5}$$

we obtain the factorization

$$V = xv + yw - z\,\frac{r^2}{2} = z\biggl(xc_x + yc_y - \frac{r^2}{2}\biggr) = z\biggl\langle \vec{r}, \vec{c} - \frac{\vec{r}}{2}\biggr\rangle = zr\biggl\langle \vec{e}_r, \vec{c} - \frac{\vec{r}}{2}\biggr\rangle.$$

Here $\vec{r} = (x, y)$ is the radius-vector of the end-point of the elastic, $\vec{e}_r = \vec{r}/r$, and $\vec{c} = (c_x, c_y)$ is the radius-vector of the centre of mass of the segment of the elastic. In Fig. 3 we have

$$\overrightarrow{OR} = \vec{r}, \quad \overrightarrow{OM} = \frac{\vec{r}}{2}, \quad \overrightarrow{OC} = \vec{c}, \quad \overrightarrow{MC} = \vec{c} - \frac{\vec{r}}{2},$$

$MP = l^{\perp}$ is the perpendicular bisector of the chord OR, and P is the projection of the centre of mass C onto the perpendicular bisector $l^{\perp}$. Therefore $PC = \langle \vec{e}_r, \vec{c} - \vec{r}/2\rangle$ is the distance from the centre of mass C to the perpendicular bisector MP. Consequently, for $z \neq 0$ the equality $V = 0$ means that the centre of mass of the segment of the elastic lies on the perpendicular bisector of the chord.

This equality is trivially satisfied for elastics centred at a vertex (see §§ 4.3, 4.4, Figs. 7, 9, 11, 13). Such elastics are fixed points of the reflections $\varepsilon^2\colon N_i \mapsto N_i$, $i = 1, 2, 3$.

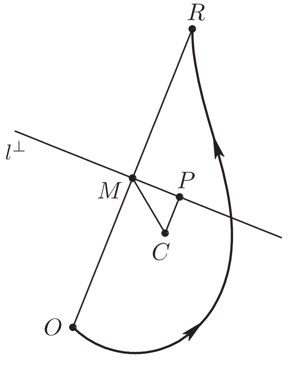

Figure 3. $V = zr\langle \vec{e}_r, \vec{c} - \vec{r}/2\rangle$

We now clarify the graphic meaning of the equation $r^2 + \rho^2 = 0$. The equation $r^2 = 0$ means that the elastic is a closed curve (the initial point coincides with the end-point, and the tangents at the initial point and the end-point are, generally speaking, different). Next, we use the expressions (5) for the centre of mass of a segment of an elastic. It is easy to see from these expressions that for $z \neq 0$ the equation $r^2 + \rho^2 = 0$ means that $c_x = c_y = 0$. In other words, the centre of mass of the segment of the elastic coincides with its initial point. Thus, the equation $r^2 + \rho^2 = 0$ means (for $z \neq 0$) that the elastic is closed and the centre of mass of its segment coincides with the initial point and the end-point. We call such elastics *remarkable*. We shall show in [7] that there exist only non-inflectional remarkable elastics (for $\lambda \in C_2$).

3.3. The invariant meaning of multiple points. We now clarify the invariant meaning of the hypersurfaces $z = 0$ and $V = 0$ and the curve $r^2 + \rho^2 = 0$ for the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$.

Remark. We point out that the nilpotent sub-Riemannian structure $(\Delta,\langle\,\cdot\,,\cdot\,\rangle)$ with the growth vector $(2, 3, 5)$ defined by the orthonormal basis (X_1, X_2) determines the left-invariant basis fields (X_1, X_2) themselves on the Lie group M uniquely up to orthogonal transformations in the plane Δ. Therefore the vector fields $(X_1, \dots, X_5)$ are determined by the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$ uniquely up to the rotations

$$\begin{aligned}(X_1, X_2, X_3, X_4, X_5) \mapsto (&\cos\varphi X_1 + \sin\varphi X_2,\ -\sin\varphi X_1 + \cos\varphi X_2,\ X_3,\\ &\cos\varphi X_4 + \sin\varphi X_5,\ -\sin\varphi X_4 + \cos\varphi X_5)\end{aligned}$$

and the reflections

$$(X_1, X_2, X_3, X_4, X_5) \mapsto (X_1, -X_2, -X_3, -X_4, X_5).$$

3.3.1. *Invariant description of the hypersurface $z = 0$.* The manifold

$$S_z = \{q \in M \mid z = 0\}$$

admits an invariant description in terms of the centre of the Lie group M

$$\begin{aligned} Z = Z(M) &= \{q \in M \mid q \cdot q' = q' \cdot q \ \ \forall\, q' \in M\} = e^{\mathbb{R}X_4} \cdot e^{\mathbb{R}X_5} \\ &= \{(x = 0,\ y = 0,\ z = 0,\ v,\ w) \mid v, w \in \mathbb{R}\} \end{aligned}$$

and the subset of M filled with the one-parameter subgroups tangent to the distribution Δ

$$e^{\mathbb{R}\Delta} = \{e^{tX} \mid X \in \Delta,\ t \in \mathbb{R}\}.$$

Proposition 3.3. *The following holds*: $S_z = Z \cdot e^{\mathbb{R}\Delta}$.

Proof. The one-parameter subgroups that are tangent to the distribution Δ can be described either as abnormal geodesics or as normal geodesics for $\lambda \in C_4, C_5, C_7$ (see [1]). Therefore,

$$e^{\mathbb{R}\Delta} = \left\{ \left(t\cos\varphi,\ t\sin\varphi,\ 0,\ \frac{t^3}{6}\sin\varphi\,,\ -\frac{t^3}{6}\cos\varphi \right) \,\middle|\, t \in \mathbb{R},\ \varphi \in S^1 \right\};$$

consequently,

$$Z \cdot e^{\mathbb{R}\Delta} = \left\{ \left(t\cos\varphi,\ t\sin\varphi,\ 0,\ \frac{t^3}{6}\sin\varphi + v,\ -\frac{t^3}{6}\cos\varphi + w \right) \right\} = \{z = 0\} = S_z.$$

Thus, the surface S_z is a certain extension of the set $e^{\mathbb{R}\Delta}$ filled with abnormal geodesics, which are straight lines in the plane (x, y). A graphic indication of this fact is that straight lines sweep out a domain of zero area: along abnormal geodesics, $z \equiv 0$.

3.3.2. *Invariant description of the hypersurface $V = 0$.* The hypersurface

$$S_V = \{q \in M \mid V = 0\}$$

can be described as a certain extension of the subset of M filled with normal geodesics corresponding to the Heisenberg case [17] ($\lambda \in C_4 \cup C_5 \cup C_6$), which are straight lines and circles in the plane (x, y). A graphic indication of this fact is that the centre of mass of a disc segment lies on the perpendicular bisector of the chord: the identity $V \equiv 0$ holds along the geodesics corresponding to $\lambda \in C_6$.

We define the following subsets of M:

$$\begin{aligned} S_V^1 &= \{q \in M \mid V = 0,\ r \neq 0\}, \\ S_V^0 &= \{q \in M \mid V = 0,\ r = 0\} = \{q \in M \mid r = 0\}, \\ K &= \{q_t \mid \dot{q}_t = \cos\theta_t\, X_1(q_t) + \sin\theta_t\, X_2(q_t),\ \ddot{\theta}_t = 0,\ q_0 = \mathrm{Id}\}, \\ K^1 &= K \setminus S_V^0, \end{aligned}$$

vector fields in M:

$$X_6 = \frac{\partial}{\partial z} + \frac{x}{2}\frac{\partial}{\partial v} + \frac{y}{2}\frac{\partial}{\partial w} = X_3 - \frac{x}{2}X_4 - \frac{y}{2}X_5,$$
$$X_7 = y\frac{\partial}{\partial v} - x\frac{\partial}{\partial w} = yX_4 - xX_5,$$

and a distribution on M:

$$L_V(q) = \operatorname{span}\bigl(X_0(q), Y(q), X_6(q), X_7(q)\bigr) \subset T_qM, \qquad q \in M.$$

We claim that the distribution L_V is invariantly determined by the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$, and the set S_V is the closure of the orbit (maximal integral manifold) of this distribution passing through any point of the set K^1, which obviously is invariantly determined by the structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$.

Lemma 3.1. *The distribution L_V is involutive.*

Proof. A straightforward calculation shows that the distribution L_V is closed with respect to the Lie bracket:

$$[X_0, Y] = 0, \qquad [X_0, X_6] = 0, \qquad [X_0, X_7] = 0,$$
$$[Y, X_6] = -2X_6, \qquad [Y, X_7] = -2X_7, \qquad [X_6, X_7] = 0.$$

Lemma 3.2. *If $q \in S_V^1$, then* $\dim L_V(q) = 4$.

Proof. We write down the basis vectors of the distribution L_V as columns with respect to the basis $\partial/\partial x, \dots, \partial/\partial w$:

$$M_V(q) := (X_0(q), Y(q), X_6(q), X_7(q)) = \begin{pmatrix} -y & x & 0 & 0 \\ x & y & 0 & 0 \\ 0 & 2z & 1 & 0 \\ -v & 3v & \dfrac{x}{2} & y \\ w & 3w & \dfrac{y}{2} & -x \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} M_V(q) = 4$ for $x^2 + y^2 \neq 0$.

Lemma 3.3. *The vector field $W = X_6$ is uniquely determined from the vector fields $X_1, \dots, X_5$ by the following conditions*:

1) $W \in \operatorname{span}(X_3, X_4, X_5)$;
2) $[W, X_1] = -\dfrac{1}{2}X_4$, $\;[W, X_2] = -\dfrac{1}{2}X_5$;
3) $W(\mathrm{Id}) = X_3(\mathrm{Id})$.

Proof. It can be verified directly that conditions 1)–3) are satisfied for the field $W = X_6$. Let us show that there are no other fields satisfying these conditions.

We expand condition 1):

$$W = aX_3 + bX_4 + cX_5, \qquad a, b, c \in C^\infty(M).$$

Using the Jacobi identity we obtain from conditions 2) the equalities $[W, X_3] = [W, X_4] = [W, X_5] = 0$. Therefore,

$$[W, X_4] = -a_v X_3 - b_v X_4 - c_v X_5 = 0,$$
$$[W, X_5] = -a_w X_3 - b_w X_4 - c_w X_5 = 0,$$

whence $a = a(x, y, z)$, $b = b(x, y, z)$, $c = c(x, y, z)$. Next,

$$[W, X_3] = -a_z X_3 - b_z X_4 - c_z X_5 = 0;$$

consequently, $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$. Finally,

$$[W, X_1] = -aX_4 - a_x X_3 - b_x X_4 - c_x X_5 = -\frac{1}{2} X_4,$$
$$[W, X_2] = -aX_5 - a_y X_3 - b_y X_4 - c_y X_5 = -\frac{1}{2} X_5,$$

whence

$$a_x = 0, \qquad a + b_x = \frac{1}{2}, \qquad c_x = 0,$$
$$a_y = 0, \qquad b_y = 0, \qquad c_y + a = \frac{1}{2}.$$

From condition 3) we obtain

$$a(0) = 1, \qquad b(0) = c(0) = 0.$$

Consequently,

$$a \equiv 1, \qquad b = -\frac{1}{2} x, \qquad c = -\frac{1}{2} y,$$

that is, $W = X_6$.

Lemma 3.4. *The vector field $W = X_7$ is uniquely determined from the vector fields $X_1, \dots, X_5$ by the following conditions*:

1) $W \in \operatorname{span}(X_4, X_5)$;
2) $[W, X_1] = X_5$, $[W, X_2] = -X_4$;
3) $W(\mathrm{Id}) = 0$.

Proof. It is easy to see that the field $W = X_7$ satisfies conditions 1)–3).

Similarly to the proof of Lemma 3.3 we expand condition 1):

$$W = aX_4 + bX_5, \qquad a, b \in C^\infty(M),$$

and obtain from condition 2) that $[W, X_3] = [W, X_4] = [W, X_5] = 0$, whence $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$. Next, it follows from condition 2) that

$$a_x = 0, \qquad a_y = 1, \qquad b_x = -1, \qquad b_y = 0,$$

and from condition 3) that $a(0, 0) = b(0, 0) = 0$. Therefore, $a = y$, $b = -x$, that is, $W = X_7$.

Lemma 3.5. *The vector field $W = Y$ is uniquely determined from the vector fields $X_1, \dots, X_5$ by the following conditions*:
1) $W \in \mathrm{Sym}(\Delta)$;
2) $[W, X_1] = -X_1$, $[W, X_2] = -X_2$;
3) $W(\mathrm{Id}) = 0$.

Proof. We use the results of [5], where the 14-dimensional Lie algebra $\mathrm{Sym}(\Delta) = \mathfrak{g}_2$ of infinitesimal symmetries of the planar nilpotent distribution with the growth vector $(2, 3, 5)$ was calculated:

$$W \in \mathrm{Sym}(\Delta) \quad \Leftrightarrow \quad W = \sum_{i=1}^{14} a_i Y_i, \quad a_i = \mathrm{const},$$

where the basis fields Y_i were described in [5], Theorem 6. Since $Y_i(\mathrm{Id}) = 0$, $i = 6, \dots, 14$, we obtain from condition 3) that

$$a_1 = \cdots = a_5 = 0.$$

Next, from the formulae for commutation of the fields Y_i with the basis fields of the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$ (see the end of the proof of Lemma 5.2 in [5]) we obtain that

$$\begin{aligned}
[W, X_1] = -X_1 \quad \Leftrightarrow \quad & a_6(-4xy^2 + 4yz) + a_7(24xy - 12z) + a_8(-432xz + 324v) \\
& \qquad - 36xa_9 - \frac{1}{3}\, y a_{10} + a_{13} + a_{14} \equiv -1, \\
& a_6(-2y^3) + a_7 18y^2 + a_8(-648u) + a_9(-54y) + 54a_{11} \equiv 0,
\end{aligned}$$

whence

$$a_6 = \cdots = a_{11} = 0, \qquad a_{13} + a_{14} = -1.$$

Next,

$$[W, X_2] = -X_2 \quad \Leftrightarrow \quad a_{12}\left(-\frac{1}{54}\right) = 0, \quad a_{14} = 1;$$

consequently,

$$a_{12} = 0, \qquad a_{13} = -2, \qquad a_{14} = 1.$$

Therefore the field $W = -2Y_{13} + Y_{14}$ is uniquely determined by conditions 1)–3).

It is easy to see that the fields $\pm X_6$, X_7, Y are uniquely determined by the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$. In Assertion 4.1 in [1] it was shown that the generator of rotations X_0 is also uniquely determined by this structure.

Corollary 3.1. *The distribution L_V is uniquely determined by the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$.*

Lemma 3.6. *The following hold*:
1) *S^1_V is a smooth 4-dimensional submanifold of M*;
2) *the equality $T_q S^1_V = L_V(q)$ holds for any $q \in S^1_V$*;
3) *the manifold S^1_V is connected.*

Proof. Part 1) follows from the fact that

$$\operatorname{grad} V = \left(v - zx,\ w - zy,\ -\frac{r^2}{2}\,,\ x,\ y \right) \neq 0 \quad \text{for} \quad r^2 \neq 0.$$

2) The fields in L_V are tangent to the manifold S_V^1, since

$$X_0 V = X_6 V = X_7 V = 0, \qquad YV = 4V,$$

and the equality $T_q S_V^1 = L_V(q)$ follows from Lemma 3.2.

3) The action of the rotations X_0 and dilatations Y takes any point $q \in S_V^1$ to a point $q' = (x', y', z', v', w')$, where $x' = 1$, $y' = 0$, $z' = 2v'$, and the whole plane $\{x = 1,\ y = 0,\ z = 2v\}$ is filled in this fashion. Therefore the connectedness of the manifold S_V^1 follows from the connectedness of this plane.

We denote by $\mathscr{O}_{\mathscr{F}}(q)$ the orbit of a point $q \in M$ under the action of the flows of the vector fields of a family $\mathscr{F} \subset \operatorname{Vec} M$:

$$\mathscr{O}_{\mathscr{F}}(q) = \bigl\{ e^{t_N f_N} \circ \dots \circ e^{t_1 f_1}(q) \mid f_i \in \mathscr{F},\ t_i \in \mathbb{R},\ N \in \mathbb{N} \bigr\},$$

and by $\mathscr{O}_{\mathscr{F}}(N)$ the orbit of the set $N \subset M$:

$$\mathscr{O}_{\mathscr{F}}(N) = \bigcup_{q \in N} \mathscr{O}_{\mathscr{F}}(q)$$

(see the description of the basic properties of an orbit in the books [9], [15]). We denote by $\operatorname{cl}(N)$ the topological closure of the set $N \subset M$.

We can now give an invariant description of the manifold S_V^1 and the set S_V.

Proposition 3.4. *The following hold*:

1) $S_V^1 = \mathscr{O}_{L_V}(q)$ *for any point* $q \in K^1$;

2) $S_V^1 = \mathscr{O}_{L_V}(K^1)$;

3) $S_V = \operatorname{cl}(S_V^1)$;

4) $S_V = \operatorname{cl}(\mathscr{O}_{L_V}(K))$.

Proof. 1) Let $q = (x, y, z, v, w)$ be any point in K^1. Then q belongs to the geodesic corresponding to a covector $\lambda \in C_6 \cup C_3 \cup C_4 \cup C_7$ and, as shown in [1], up to rotations and dilatations,

$$x = \sin t, \qquad y = 1 - \cos t, \qquad z = \frac{t - \sin t}{2}\,,$$
$$v = \frac{\cos 2t - 4\cos t + 3}{4}\,, \qquad w = \frac{\sin 2t - 4\sin t + 2t}{4}$$

or

$$x = t, \qquad y = z = v = 0, \qquad w = -\frac{t^3}{6}\,.$$

A straightforward calculation shows that $V(q) = 0$; therefore, $q \in S_V^1$.

The inclusion $S_V^1 \subset \mathscr{O}_{L_V}(q)$ follows from the connectedness of S_V^1 and the equality $T_q S_V^1 = L_V(q)$.

We now prove the reverse inclusion $\mathscr{O}_{L_V}(q) \subset S_V^1$. Arguing by contradiction, suppose that there exists a point $q_1 \in \mathscr{O}_{L_V}(q) \setminus S_V^1$. Then either $q_1 \in S_V^0$ or $V(q_1) \neq 0$. The inclusion $q_1 \in S_V^0$ is impossible, since the set S_V^0 is invariant under the flows of the fields in L_V. The inequality $V(q_1) \neq 0$ is impossible, since the level surface $\{V(q) = 0\}$ is also invariant under the fields in L_V. The inclusion $\mathscr{O}_{L_V}(q) \subset S_V^1$ and therefore the equality $\mathscr{O}_{L_V}(q) = S_V^1$ are proved.

Part 2) immediately follows from part 1).

3) It follows from the inclusion $S_V^1 \subset S_V$ that $\mathrm{cl}(S_V^1) \subset \mathrm{cl}(S_V) = S_V$. We now prove the reverse inclusion $S_V \subset \mathrm{cl}(S_V^1)$. In view of the decomposition $S_V = S_V^1 \cup S_V^0$ it is sufficient to prove the inclusion $S_V^0 \subset \mathrm{cl}(S_V^1)$.

Let $q = (0, 0, z, v, w) \in S_V^0$. If $\rho^2 = v^2 + w^2 \neq 0$, then a sequence of points of the form

$$(x', y', z', v', w'), \qquad x' = r' \cos\chi', \quad y' = r' \sin\chi', \quad \cos(\chi' - \omega) = \frac{z}{2\rho}\, r', \quad r' \to 0,$$

converges to the point q and belongs to the manifold S_V^1. If $\rho^2 = 0$, then for such points one can take points of the form

$$(x', 0, z, v', 0), \qquad v' = \frac{z}{2}\, x', \quad x' \to 0.$$

We have proved the inclusions $S_V^0 \subset \mathrm{cl}(S_V^1)$, $S_V \subset \mathrm{cl}(S_V^1)$ and the equality $S_V^1 = \mathrm{cl}(S_V)$.

4) We obtain from the preceding parts the equalities

$$\begin{aligned} S_V &= \mathrm{cl}(\mathscr{O}_{L_V}(q)) \qquad \forall\, q \in K^1, \\ S_V &= \mathrm{cl}(K^1). \end{aligned}$$

But $K = K^1 \cup S_V^0$; therefore,

$$\mathscr{O}_{L_V}(K) = \mathscr{O}_{L_V}(K^1) \cup \mathscr{O}_{L_V}(S_V^0) = \mathscr{O}_{L_V}(K^1) \cup S_V^0.$$

It follows from the inclusion $S_V^0 \subset S_V$ that $S_V = \mathrm{cl}(\mathscr{O}_{L_V}(K))$.

3.3.3. *Invariant description of the curve* $r^2 + \rho^2 = 0$. The curve

$$S_{r^2+\rho^2} = \{q \in M \mid r^2 + \rho^2 = 0\}$$

admits the following simple invariant description: it is the trajectory of the field

$$X_3 = [X_1, X_2] = \frac{\partial}{\partial z} + x\,\frac{\partial}{\partial v} + y\,\frac{\partial}{\partial w}$$

passing through the initial point q_0 — the identity element Id of the Lie group M.

Proposition 3.5. *We have the equality* $S_{r^2+\rho^2} = e^{\mathbb{R}X_3}(\mathrm{Id})$.

Proof. Since $X_3\big|_{S_{r^2+\rho^2}} = \partial/\partial z$, the field X_3 is tangent to the curve $S_{r^2+\rho^2}$ and

$$e^{\mathbb{R}X_3}(\mathrm{Id}) = \{(0, 0, z, 0, 0) \mid z \in \mathbb{R}\} = S_{r^2+\rho^2}.$$

It follows from the remark at the beginning of § 3.3 that the curve $S_{r^2+\rho^2}$ is invariantly determined by the sub-Riemannian structure $(\Delta, \langle\,\cdot\,,\cdot\,\rangle)$.

§ 4. Fixed points of symmetries in the inverse image of the exponential map

4.1. Fixed points of rotations in N. It is obvious that the rotations

$$e^{s\vec{h}_0}\colon (\theta, c, \alpha, \beta, t) \mapsto (\theta + s, c, \alpha, \beta + s, t) \tag{6}$$

have no fixed points in N for $s \neq 2\pi n$, $n \in \mathbb{N}$.

4.2. Fixed points of reflections in N_1. We shall be using the elliptic coordinates k, φ, and ψ, as well as the polar coordinates α, β in the initial cylinder C (see [1], [6]). For that we need the Jacobi elliptic functions cn, sn (see [18], [19]).

Let $\nu = (k, \varphi, \alpha, \beta, t) \in N_1$; then by Proposition 4.1 in [6] we have

$$\nu^i = \varepsilon^i(\nu) = (k, \varphi^i, \alpha, \beta^i, t) \in N_1.$$

The fixed points of reflections in the domain N_1 can be expressed in terms of the following invariant of the two-parameter symmetry group $G_{\vec{h}_0, Z}$:

$$\tau = \frac{\sqrt{\alpha}\,(\varphi + \varphi_t)}{2} = \sqrt{\alpha}\,\varphi + \frac{\delta}{2}\,.$$

Theorem 4.1. *Let $\nu \in N_1$. The following hold*:

1) $e^{\sigma\vec{h}_0}(\nu^1) = \nu \;\;\Leftrightarrow\;\; \operatorname{cn}\tau = 0, \;\; \sigma = 0$;
2) $e^{\sigma\vec{h}_0}(\nu^2) = \nu \;\;\Leftrightarrow\;\; \operatorname{sn}\tau = 0, \;\; \sigma = 2\beta$;
3) $e^{\sigma\vec{h}_0}(\nu^3) = \nu$ *is impossible.*

Proof. In the elliptic coordinates the equality $e^{\sigma\vec{h}_0}(\nu^i) = \nu$ takes the form

$$e^{\sigma\vec{h}_0}(\nu^i) = (k, \varphi^i, \alpha, \beta^i + \sigma, t) = (k, \varphi, \alpha, \beta, t) = \nu,$$

which is equivalent to the equalities

$$\varphi^i = \varphi, \qquad \beta^i + \sigma = \beta. \tag{7}$$

Part 1). According to Proposition 4.1 in [6] the equalities (7) can be rewritten in the form

$$\varphi + \varphi_t = \frac{2K}{\sqrt{\alpha}} \left(\operatorname{mod} \frac{4K}{\sqrt{\alpha}}\right), \qquad \sigma = 0,$$

where K is the complete elliptic integral of the second kind [18], [19], which is equivalent to

$$\tau = K \ (\operatorname{mod} 2K), \qquad \sigma = 0,$$

that is,

$$\operatorname{cn}\tau = 0, \qquad \sigma = 0.$$

Part 1) of the proposition is proved; the other two parts can be proved in similar fashion.

Remark. We point out the graphic meaning of the fixed points of the reflections $\varepsilon^i\colon N_1 \to N_1$ for the standard pendulum in the plane (θ, c) and the inflectional Euler elastics in the plane (x, y).

1) The equality $\operatorname{cn}\tau = 0$ is equivalent to the equality $c = 0$ — these are the inflection points of elastics (zeros of the curvature c). See Figs. 4, 5.

2) The equality $\operatorname{sn}\tau = 0$ is equivalent to the equality $\theta = 0$ — these are the vertices of elastics (extrema of the curvature c). See Figs. 6, 7.

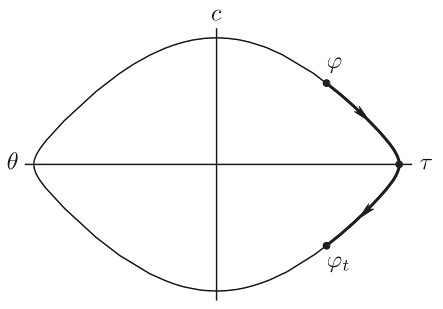

Figure 4. $\operatorname{cn}\tau = 0, \ \ \nu \in N_1$

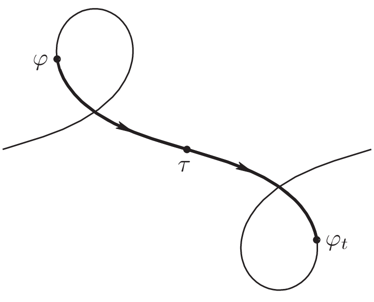

Figure 5. An inflectional elastic with centre at an inflection point

4.3. Fixed points of reflections in N_2. Let $\nu = (k, \psi, \alpha, \beta, t) \in N_2$; then by Proposition 4.1 in [6] we have

$$\nu^i = \varepsilon^i(\nu) = (k, \psi^i, \alpha, \beta^i, t) \in N_2.$$

In the domain N_2 we consider the following invariant of the group $G_{\vec{h}_0,Z}$:

$$\tau = \frac{\sqrt{\alpha}\,(\psi + \psi_t)}{2} = \sqrt{\alpha}\,\psi + \frac{\delta}{2k}\,.$$

Theorem 4.2. *Let $\nu \in N_2$. The following hold*:

1) $e^{\sigma\vec{h}_0}(\nu^1) = \nu$ *is impossible*;
2) $e^{\sigma\vec{h}_0}(\nu^2) = \nu \ \Leftrightarrow \ \operatorname{sn}\tau\operatorname{cn}\tau = 0, \ \ \sigma = 2\beta$;
3) $e^{\sigma\vec{h}_0}(\nu^3) = \nu$ *is impossible.*

Proof. First we consider parts 1), 3). If $i = 1, 3$, then we obtain from Proposition 4.1 in [6] that

$$\nu \in N_2^{\pm} \quad \Rightarrow \quad \nu^i, e^{\sigma\vec{h}_0}(\nu^i) \in N_2^{\mp};$$

therefore the equality $e^{\sigma\vec{h}_0}(\nu_i) = \nu$ is impossible.

Part 2). We have

$$\nu \in N_2^{\pm} \quad \Rightarrow \quad \nu^2, e^{\sigma\vec{h}_0}(\nu^2) \in N_2^{\pm}$$

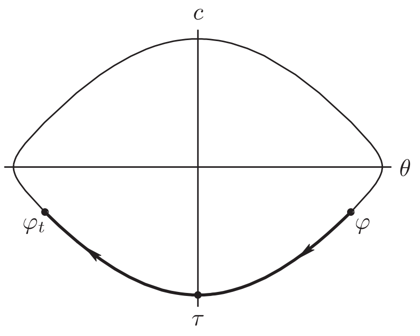

Figure 6. $\operatorname{sn}\tau = 0$, $\nu \in N_1$

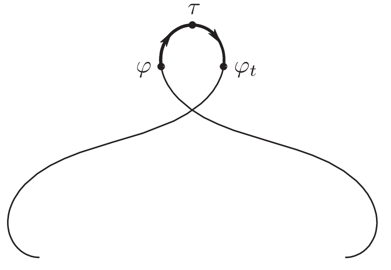

Figure 7. An inflectional elastic with centre at a vertex

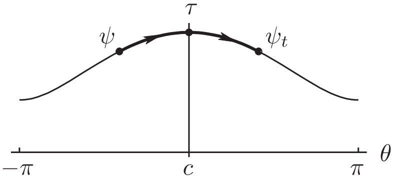

Figure 8. $\operatorname{sn}\tau = 0$, $|c| = \max$, $\nu \in N_2$

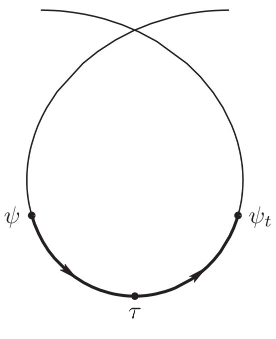

Figure 9. A non-inflectional elastic with centre at a vertex

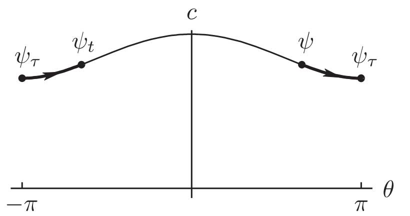

Figure 10. $\operatorname{sn}\tau = 0$, $|c| = \min$, $\nu \in N_2$

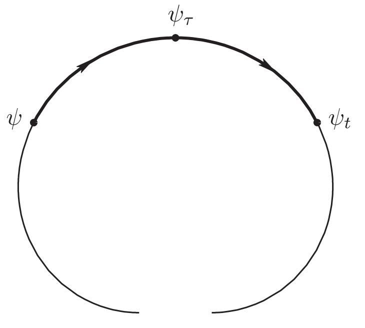

Figure 11. A non-inflectional elastic with centre at a vertex

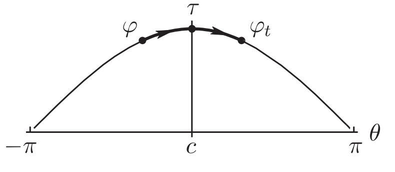

Figure 12. $\tau = 0$, $\nu \in N_3$

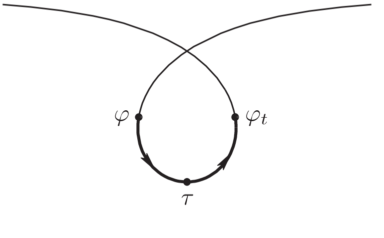

Figure 13. A critical elastic with centre at a vertex

and the equality $e^{\sigma \vec{h}_0}(\nu^2) = \nu$ in the elliptic coordinates takes the form

$$\psi + \psi_t = 0 \left(\operatorname{mod} \frac{2K}{\sqrt{\alpha}} \right), \qquad \sigma = 2\beta,$$

which is equivalent to

$$\tau = 0 \ (\operatorname{mod} K), \qquad \sigma = 2\beta,$$

that is,

$$\operatorname{sn} \tau \operatorname{cn} \tau = 0, \qquad \sigma = 2\beta.$$

Remark. We point out the graphic meaning of the fixed points of the reflections $\varepsilon^2 \colon N_2 \to N_2$. The equality $\operatorname{sn} \tau \operatorname{cn} \tau = 0$ is equivalent to the equalities $\theta = 0 \ (\operatorname{mod} \pi)$, $|c| = \max, \min$ — these are the vertices of non-inflectional elastics (extrema of the curvature c). See Figs. 8–11.

There do not exist fixed points of $\varepsilon^1 \colon N_2 \to N_2$, since the elastics corresponding to N_2 have no inflection points; this is why they are called non-inflectional.

4.4. Fixed points of reflections in N_3. Let $\nu = (\varphi, \alpha, \beta, t) \in N_3$; then

$$\nu^i = \varepsilon^i(\nu) = (\varphi^i, \alpha, \beta^i, t) \in N_3.$$

On the set N_3, the invariant τ is obtained by passing to the limit as $k \to 1 - 0$ from both domains N_1, N_2:

$$\tau = \frac{\sqrt{\alpha}\,(\varphi + \varphi_t)}{2} = \sqrt{\alpha}\,\varphi + \frac{\delta}{2}\,.$$

Theorem 4.3. *Let $\nu \in N_3$. The following hold*:

1) $e^{\sigma \vec{h}_0}(\nu^1) = \nu$ *is impossible*;
2) $e^{\sigma \vec{h}_0}(\nu^2) = \nu \ \Leftrightarrow \ \tau = 0, \ \ \sigma = 2\beta$;
3) $e^{\sigma \vec{h}_0}(\nu^3) = \nu$ *is impossible.*

Proof. The proof is similar to the proof of Theorem 4.2.

Remark. The graphic meaning of the fixed points of the reflections $\varepsilon^2 \colon N_3 \to N_3$: the equality $\tau = 0$ means that $\theta = 0, \ \ |c| = \max$ — these are the vertices of critical elastics (extrema of the curvature c). See Figs. 12, 13.

There do not exist fixed points of $\varepsilon^1 \colon N_3 \to N_3$, since critical elastics have no inflection points.

4.5. Fixed points of reflections in N_6.

Theorem 4.4. *Let $\nu = (\theta, c, \alpha, \beta, t) \in N_6$. The following hold*:

1) $e^{\sigma \vec{h}_0}(\nu^1) = \nu$ *is impossible*;
2) $e^{\sigma \vec{h}_0}(\nu^2) = \nu \ \Leftrightarrow \ \sigma = 2\theta + ct \ (\operatorname{mod} 2\pi)$;
3) $e^{\sigma \vec{h}_0}(\nu^3) = \nu$ *is impossible.*

Proof. Let $\nu = (\theta, c, \alpha, \beta, t) \in N_6$. If $i = 1, 3$, then

$$\nu \in C_6^{\pm} \quad \Rightarrow \quad \nu^i, e^{\sigma \vec{h}_0}(\nu^i) \in C_6^{\mp};$$

therefore the equality $e^{\sigma\vec{h}_0}(\nu_i) = \nu$ is impossible. Next,

$$\nu \in C_6^{\pm} \quad \Rightarrow \quad \nu^2, e^{\sigma\vec{h}_0}(\nu^2) \in C_6^{\pm}$$

and the equality

$$e^{\sigma\vec{h}_0}(\nu^2) = e^{\sigma\vec{h}_0}(-\theta_t, c_t, t) = (\sigma - \theta - ct, c, t) = (\theta, c, t) = \nu$$

is equivalent to the equality $\sigma = 2\theta + ct$.

§ 5. General description of the Maxwell strata MAX_i

We summarize the analysis of the Maxwell strata corresponding to rotations and reflections. We obtain the following assertions from the results of §§ 3, 4.

Theorem 5.1. *Let $\nu = (\lambda, t) \in N$. Then*

$$\nu \in \mathrm{MAX}_0 \quad \Leftrightarrow \quad r_t = \rho_t = 0.$$

Proof. The assertion follows from Proposition 3.2 by the fact that the rotations (6) have no fixed points in N.

Theorem 5.2. *Let $\nu = (\lambda, t) \in N_1$. Then*

$$\nu \in \mathrm{MAX}_1 \qquad \Leftrightarrow \qquad \begin{cases} \text{a) } z_t = 0, \quad \operatorname{cn} \tau \neq 0, \quad \tau = \sqrt{\alpha}\, \frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b) } q_t = q_0. \end{cases}$$

Remark. In the subsequent paper [7] we shall prove that case b) of Theorem 5.2 is not realized.

Proof. First we consider the case $\alpha = 1$, $\beta = 0$. Recall that by the definition of the Maxwell stratum the inclusion $\nu = (\lambda, t) \in \mathrm{MAX}_1$ means that for some $\sigma \in \mathbb{R}$ we have

$$e^{\sigma\vec{h}_0}(\nu^1) \neq \nu, \qquad e^{\sigma X_0}(q_t^1) = q_t.$$

Conditions under which the above inequality and equality hold were found in Proposition 3.1 and Theorem 4.1.

First let $r_t^2 + \rho_t^2 > 0$. Suppose that $\nu \in \mathrm{MAX}_1$. Then $z_t = 0$ and $\sigma = 0$ (see Theorem 4.1). By Proposition 3.1 we have $\operatorname{cn} \tau \neq 0$. Conversely, if $z_t = 0$ and $\operatorname{cn} \tau \neq 0$, we choose $\sigma = 0$ and obtain $\nu \in \mathrm{MAX}_1$.

Let $r_t^2 + \rho_t^2 = 0$. If $\nu \in \mathrm{MAX}_1$, then by Theorem 4.1 we obtain $z_t = 0$; hence $q_t = 0 = q_0$. Conversely, if $z_t = 0$, then we take any $\sigma \neq 0$ and obtain $\nu \in \mathrm{MAX}_1$.

In the general case the assertion follows from the special case $\alpha = 1$, $\beta = 0$ due to the invariance of the Maxwell stratum MAX_1 under rotations and dilatations.

Theorem 5.3. *Let $\nu = (\lambda, t) \in N_1$. Then*

$$\nu \in \mathrm{MAX}_2 \qquad \Leftrightarrow \qquad \begin{cases} \text{a) } V_t = 0, \quad \operatorname{sn} \tau \neq 0, \quad \tau = \sqrt{\alpha}\, \frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b) } r_t = \rho_t = 0. \end{cases}$$

Proof. As in the preceding theorem, it is sufficient to consider only the case $\alpha = 1$, $\beta = 0$, and use in this case Proposition 3.1 and Theorem 4.1.

Necessity. Let $\nu \in \mathrm{MAX}_2$. Then $V_t = 0$. We now prove that $\operatorname{sn}\tau \neq 0$ or $r_t = \rho_t = 0$.

Let $r_t > 0$; we prove that $\operatorname{sn}\tau \neq 0$. Arguing by contradiction, suppose that $\operatorname{sn}\tau = 0$. Then from the parametrization of the geodesics [1] we obtain $y_t = 0$. Therefore, $\chi_t = 0 \pmod{\pi}$; consequently, $\sigma = 2\chi_t = 0 \pmod{2\pi} = 2\beta$. But then $\operatorname{sn}\tau \neq 0$, a contradiction.

Let $r_t = 0$, $\rho_t > 0$; then $\sigma = 2\omega_t - \pi$. If $\operatorname{sn}\tau = 0$, then from the explicit formulae for the geodesics [1] we obtain $x_t|_{\operatorname{sn}\tau=0} = 2(2\mathrm{E}(p) - p) = 0$, where $\mathrm{E}(p)$ is a Jacobi elliptic function, $p = t/2$, and $v_t = 0$. This means that $\omega_t = \pi/2 \pmod{\pi}$; therefore, $\sigma = 0 \pmod{2\pi} = 2\beta$. Consequently, again $\operatorname{sn}\tau \neq 0$, a contradiction. The necessity is proved.

Sufficiency. Let $r_t^2 + \rho_t^2 \neq 0$, $V_t = 0$, and $\operatorname{sn}\tau \neq 0$. Then, choosing $\sigma = 2\chi_t$ (for $r_t > 0$) or $\sigma = 2\omega_t - \pi$ (for $r_t = 0$, $\rho_t > 0$), we verify that $\nu \in \mathrm{MAX}_2$.

Theorem 5.4. *Let* $\nu = (\lambda, t) \in N_1$. *Then*

$$\nu \in \mathrm{MAX}_3 \cap N_1 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

Proof. The assertion follows from Proposition 3.1, Theorem 4.1, and the invariance of the Maxwell stratum MAX_3 under the group $G_{\vec{h}_0, Z}$.

Theorem 5.5. *Let* $\nu = (\lambda, t) \in N_2$. *Then*

$$\nu \in \mathrm{MAX}_1 \quad \Leftrightarrow \quad z_t = 0.$$

Proof. The proof is similar to the proof of Theorem 5.4.

Theorem 5.6. *Let* $\nu = (\lambda, t) \in N_2$. *Then*

$$\nu \in \mathrm{MAX}_2 \qquad \Leftrightarrow \qquad \begin{cases} \text{a) } V_t = 0, \quad \operatorname{sn}\tau\operatorname{cn}\tau \neq 0, \quad \tau = \sqrt{\alpha}\,\frac{\psi+\psi_t}{2} \\ \textit{or} \\ \text{b) } r_t = \rho_t = 0. \end{cases}$$

Proof. As before, we consider only the case $\alpha = 1$, $\beta = 0$.

Let $r_t > 0$.

Necessity: if $\nu \in \mathrm{MAX}_2$, then $V_t = 0$ and $\sigma = 2\chi_t$. If $\operatorname{sn}\tau\operatorname{cn}\tau = 0$, then $y_t = 0$; therefore $\chi_t = \pi n$ and $\sigma = 2\pi n = 2\beta \pmod{2\pi}$; a contradiction.

Sufficiency: if $V_t = 0$ and $\operatorname{sn}\tau\operatorname{cn}\tau \neq 0$, then we choose $\sigma = 2\chi_t$ and obtain $\nu \in \mathrm{MAX}_2$.

Let $r_t = 0$ and $\rho_t > 0$.

Necessity: if $\nu \in \mathrm{MAX}_2$, then $V_t = 0$ and $\sigma = 2\omega_t - \pi$. If $\operatorname{sn}\tau\operatorname{cn}\tau = 0$, then $v_t = 0$; therefore $\omega_t = \pi/2 \pmod{\pi}$ and $\sigma = 0 \pmod{2\pi}$; a contradiction.

Sufficiency: if $V_t = 0$ and $\operatorname{sn}\tau\operatorname{cn}\tau \neq 0$, then we set $\sigma = 2\omega_t - \pi$ and obtain $\nu \in \mathrm{MAX}_2$.

Let $r_t = \rho_t = 0$. Then $V_t = 0$ and choosing any $\sigma \neq 0$ we obtain $\nu \in \mathrm{MAX}_2$.

Theorem 5.7. *Let* $\nu = (\lambda, t) \in N_2$. *Then*

$$\nu \in \mathrm{MAX}_3 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

Proof. The proof is similar to the proof of Theorem 5.4.

Theorem 5.8. *Let* $\nu = (\lambda, t) \in N_3$. *Then*

$$\nu \in \mathrm{MAX}_1 \quad \Leftrightarrow \quad z_t = 0.$$

Proof. The proof is similar to the proof of Theorem 5.4.

Theorem 5.9. *Let* $\nu = (\lambda, t) \in N_3$. *Then*

$$\nu \in \mathrm{MAX}_2 \qquad \Leftrightarrow \qquad \begin{cases} \text{a) } V_t = 0, \quad \tau \neq 0, \quad \tau = \sqrt{\alpha}\,\frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b) } r_t = \rho_t = 0. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 5.6.

Theorem 5.10. *Let* $\nu = (\lambda, t) \in N_3$. *Then*

$$\nu \in \mathrm{MAX}_3 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

Proof. The proof is similar to the proof of Theorem 5.4.

Remark. We point out the graphic meaning of the description of the Maxwell strata in Theorems 5.1–5.10.

The equalities $r_t = \rho_t = 0$ defining the stratum MAX_0 (Theorem 5.1) mean that the elastic is closed ($x_t = y_t = 0$), while the centre of mass of its segment coincides with the initial point ($c_x = c_y = 0$). The rotation of such an elastic through any angle $s \neq 2\pi n$ produces a new elastic with the same end-point, area, and centre of mass.

Modulo rotations, the reflection ε^2 of the elastic in the perpendicular bisector of the chord $l^\perp$ acts on its end-point (x, y), area z, and centre of mass (c_x, c_y) as follows:

$$\varepsilon^2 \colon (x, y, z, c_x, c_y) \mapsto (x, y, z, c_x^2, c_y^2), \tag{8}$$

where $\varepsilon^2 \colon (c_x, c_y) \mapsto (c_x^2, c_y^2)$ is the reflection of the centre of mass in the perpendicular bisector $l^\perp$. Recall that the equality $V = 0$ means that $(c_x, c_y) \in l^\perp$, that is, $(c_x^2, c_y^2) = (c_x, c_y)$; and the equalities $\operatorname{sn}\tau = 0$, $\nu \in N_1$; $\operatorname{sn}\tau \operatorname{cn}\tau = 0$, $\nu \in N_2$; $\tau = 0$, $\nu \in N_3$, mean that the equality $V = 0$ is trivially valid when the elastic is centred at a vertex. Therefore the condition of part a) of Theorems 5.3, 5.6, 5.9 means that the centre of mass of the segment of the elastic lies on the perpendicular bisector of the chord and the elastic is centred at a non-vertex. It is obvious that the reflection in the perpendicular bisector takes such an elastic to another elastic with the same initial point, end-point, area, and centre of mass (the map (8) becomes the identity map); this is a point of the stratum MAX_2. The condition of part b) of Theorems 5.3, 5.6, 5.9 determines a point of the stratum MAX_0.

The reflection ε^1 of the elastic in the centre of the chord l acts on its end-point (x, y) and area z as follows:

$$\varepsilon^1 \colon (x, y, z) \mapsto (x, y, -z);$$

Therefore the equality $z_t = 0$ (Theorems 5.2, 5.4, 5.5, 5.7, 5.8, 5.10) must necessarily hold at the points of the strata MAX_1, MAX_3. We could not complement this argument by an analysis of the location of the centre of mass (as above, for the strata MAX_0, MAX_2), since for $z = 0$ the segment of an elastic has no finite centre of mass — in this case the centre of mass goes away to infinity or is not defined at all.

Theorem 5.11. *We have* $\mathrm{MAX}_i \cap N_j = \varnothing$, $i = 0, 1, 2, 3$, $j = 4, 5, 7$.

Proof. The geodesics $q_s = \mathrm{Exp}(\lambda, s)$ corresponding to $\lambda \in C_4 \cup C_5 \cup C_7$ are optimal on the whole ray $s \in [0, +\infty)$ (see [1]); therefore they do not contain Maxwell points.

Theorem 5.12. *We have* $\mathrm{MAX}_i \cap N_6 = \varnothing$, $i = 0, 1, 2, 3$.

Proof. Since the Maxwell strata are invariant under the action of the group $G_{\vec{h}_0, Z}$, it is sufficient to consider only the case $c = 1$, $\theta = 0$. Then the geodesic is parametrized as follows [1]:

$$x_t = \sin t, \qquad y_t = 1 - \cos t, \qquad z_t = \frac{t - \sin t}{2},$$
$$v_t = \frac{\cos 2t - 4\cos t + 3}{4}, \qquad w_t = \frac{\sin 2t - 4\cos t + 2t}{4}.$$

0) Let $t > 0$; we claim that $r_t^2 + \rho_t^2 \neq 0$. If $r_t^2 = 0$, then $t = 2\pi n$, $n \in \mathbb{N}$. But then $w_t = \pi n \neq 0$. Thus, $r_t^2 + \rho_t^2 \neq 0$; therefore $\mathrm{MAX}_0 \cap N_6 = \varnothing$.

1) We have $z_t \neq 0$ for $t > 0$; consequently, $\mathrm{MAX}_1 \cap N_6 = \varnothing$.

2.a) Let $\nu \in \mathrm{MAX}_2 \cap N_6$ and $r_t^2 > 0$; then $\sigma = 2\chi_t = t$. But from the inequality $e^{\sigma \vec{h}_0}(\nu^2) \neq \nu$ we obtain that $\sigma \neq 2\theta + ct = t$; a contradiction.

2.b) Let $\nu \in \mathrm{MAX}_2 \cap N_6$ and $r_t = 0$, $\rho_t > 0$. Then $t = 2\pi n$, $n \in \mathbb{N}$. Therefore, $v_t = 0$, $w_t = 0$, $\omega_t = \pi/2 \pmod{\pi}$, $\sigma = 2\omega_t - \pi = t \pmod{2\pi}$. Consequently, $e^{\sigma \vec{h}_0}(\nu^2) = \nu$; a contradiction.

2.c) As shown in part 0), $r_t^2 + \rho_t^2 \neq 0$ for $\nu \in N_6$. Thus, $\mathrm{MAX}_2 \cap N_6 = \varnothing$.

3) We have $z_t \neq 0$ for $t > 0$; consequently, $\mathrm{MAX}_3 \cap N_6 = \varnothing$.

We now summarize the present paper: we put together all the results on the Maxwell strata obtained in Theorems 5.1–5.12.

Theorem 5.13. 1) *Let* $\nu \in N_1$. *Then*

$$1.0)\quad \nu \in \mathrm{MAX}_0 \quad \Leftrightarrow \quad r_t = \rho_t = 0;$$

$$1.1)\quad \nu \in \mathrm{MAX}_1 \quad \Leftrightarrow \quad \begin{cases} \text{a)} \quad z_t = 0, \ \operatorname{cn} \tau \neq 0, \ \tau = \sqrt{\alpha}\, \frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b)} \quad q_t = q_0; \end{cases}$$

$$1.2)\quad \nu \in \mathrm{MAX}_2 \quad \Leftrightarrow \quad \begin{cases} \text{a)} \quad V_t = 0, \ \operatorname{sn} \tau \neq 0, \ \tau = \sqrt{\alpha}\, \frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b)} \quad r_t = \rho_t = 0; \end{cases}$$

$$1.3)\quad \nu \in \mathrm{MAX}_3 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

2) *Let* $\nu \in N_2$. *Then*

$$2.0)\quad \nu \in \mathrm{MAX}_0 \quad \Leftrightarrow \quad r_t = \rho_t = 0;$$
$$2.1)\quad \nu \in \mathrm{MAX}_1 \quad \Leftrightarrow \quad z_t = 0;$$

$$2.2)\quad \nu \in \mathrm{MAX}_2 \quad \Leftrightarrow \quad \begin{cases} \text{a)} \quad V_t = 0, \ \operatorname{sn} \tau \operatorname{cn} \tau \neq 0, \ \tau = \sqrt{\alpha}\, \frac{\psi + \psi_t}{2} \\ \textit{or} \\ \text{b)} \quad r_t = \rho_t = 0; \end{cases}$$

$$2.3)\quad \nu \in \mathrm{MAX}_3 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

3) *Let* $\nu \in N_3$. *Then*

$$3.0)\quad \nu \in \mathrm{MAX}_0 \quad \Leftrightarrow \quad r_t = \rho_t = 0;$$
$$3.1)\quad \nu \in \mathrm{MAX}_1 \quad \Leftrightarrow \quad z_t = 0;$$

$$3.2)\quad \nu \in \mathrm{MAX}_2 \quad \Leftrightarrow \quad \begin{cases} \text{a)} \quad V_t = 0, \ \tau \neq 0, \ \tau = \sqrt{\alpha}\, \frac{\varphi + \varphi_t}{2} \\ \textit{or} \\ \text{b)} \quad r_t = \rho_t = 0; \end{cases}$$

$$3.3)\quad \nu \in \mathrm{MAX}_3 \quad \Leftrightarrow \quad z_t = V_t = 0.$$

4) *We have* $\mathrm{MAX}_i \cap N_j = \varnothing$, $i = 0, \dots, 3$, $j = 4, \dots, 7$.

The study of solubility of the equations defining the Maxwell strata, as well as localization of their roots will be the contents of the subsequent paper [7].

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