Maxwell Strata and Conjugate Points in Sub-Riemannian Problem on Group SH(2)

Yasir Awais Butt, Yuri L. Sachkov, Aamer Iqbal Bhatti

Muhammad Ali Jinnah University, Islamabad, Pakistan
Program Systems Institute of Russian Academy of Sciences, Pareslavl Zalessky, Russia
Muhammad Ali Jinnah University, Islamabad, Pakistan

yasir_awais2000@yahoo.com, sachkov@sys.botik.ru, aib@jinnah.edu.pk

Sub-Riemannian geometry has experienced resurgence of interest and extensive research for past several decades. It has emerged as an extremely rich framework with a unique character seeking applications in various fields of pure and applied mathematics such as classical and quantum mechanics, control theory, geometric analysis, stochastic calculus and evolution equations. The renewed interest is also attributed to the fact that sub-Riemannian geometry has given entirely new and richer perspective to some older problems such as image inpainting, neurophysiology of vision and quantum control [1]. Consequently, research in sub-Riemannian problems via geometric control methods on various Lie groups such as the Heisenberg group, S^3, SL(2), SU(2), SE(2), Engel group etc. has been paticularly popular for two decades now. From control theory perspective, sub-Riemannian geometry models optimal control problems for nonholonomic systems such as motion planning and control of robots, falling cats, parking of cars, rolling of bodies on plane without sliding, satellites, vision, quantum phases and even finance. Magnificence of sub-Riemannian geometry as an optimal control framework drew our attention to the sub-Riemannian problem on the group of motions of pseudo Euclidean plane. The pseudo Euclidean plane F_1^{1+1} is $(1+1)$-dimensional space defined over field of real numbers $\mathbb{R}$ and endowed with a non-degenerate indefinite quadratic form q:

$$q(x) = x_1^2 - x_2^2.$$

The motions of pseudo Euclidean plane are distance and orientation preserving maps of the points in the plane. The motions describe the hyperbolic roto-translations of the pseudo Euclidean plane and form a 3-dimensional Lie group known as special hyperbolic group SH(2) [2]. The driftless control system on SH(2) is described as follows:

$$\dot{q} \;=\; u_1 f_1(q) + u_2 f_2(q), \qquad q \in M = \mathrm{SH}(2), \qquad (u_1, u_2) \in \mathbb{R}^2. \tag{1}$$

Here, (1) is the control system with bounded inputs u_i and control distribution $\Delta = \mathrm{span}\{f_1, f_2\}$. The vector fields f_i satisfy the Lie bracket relations:

$$[f_2, f_1] = f_0, \qquad [f_1, f_0] = 0, \qquad [f_2, f_0] = f_1.$$

The sub-Riemannian problem on control system (1) is defined as:

$$q(0) = Id, \qquad q(t_1) = q_1, \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{3}$$

In (2), $q(0)$ and $q(t_1)$ represent the initial and the final states whereas l (3) is the sub-Riemannian distance (length functional) to be minimized. In coordinates $q = (x, y, z)$, the control system (1) is given as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cosh z \\ \sinh z \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2. \tag{4}$$

We applied the Pontryagin Maximum Principle (PMP) on (1)–(3) to calculate the extremal controls $\tilde{u}(t)$ and the extremal trajectories. Since the problem is 3D contact, there are no nontrivial abnormal trajectories. A change of coordinates in the vertical subsystem of the normal Hamiltonian system transforms it into a mathematical pendulum. The phase cylinder C of the pendulum is decomposed into five connected subsets $C_i \quad i = 1, \ldots, 5$ depending upon the energy of the pendulum. Suitable elliptic coordinates i.e. reparametrized energy k and reparametrized time φ are introduced on each C_i and such that the flow of the vertical subsystem is rectified. Computation of the Hamiltonian flow then follows from integration of vertical and horizontal subsystem and the resulting extremal trajectories are parametrized by Jacobi elliptic functions. Further analysis/simulations reveal the qualitative nature of extremal trajectories.

Parametrization of extremal trajectories is followed by second order optimality analysis based on description of Maxwell strata and conjugate loci. Since the vertical subsystem is a mathematical pendulum, it admits reflection symmetries in the phase portrait which are used to obtain complete description of Maxwell strata. The fixed points of the extremals λ in the preimage and the multiple points in the image of exponential mapping are used to obtain complete description of the Maxwell strata and compute the first Maxwell time t_1^{MAX} for $\lambda \in C_i$, $i = 1, \ldots, 5$. On the basis of Maxwell strata and Maxwell time, we obtain a global upper bound on cut time in the sub-Riemannian problem on SH(2) which happens to be the first Maxwell time t_1^{MAX}. We then turn to the problem of characterizing the conjugate points. Computation and simplification of Jacobian for $\lambda \in C_1 \cup C_2$ reveals a rather unexpected symmetry with respect to bounds of conjugate times in these cases which hasn't been observed in corresponding analysis in sub-Riemannian problem on SE(2) [3], Engel group [5] and Euler Elasticae problem [4]. It turns out that the first conjugate time $t_1^{C_1}$ for $\lambda \in C_1$ is bounded as $4K(k) \leq t_1^{C_1} \leq 2p_1^1(k)$ where $p_1^1(k)$ is the first root of a function $f_1(p) = [\operatorname{cn} p \, \mathrm{E}(p) - \operatorname{sn} p \operatorname{dn} p]$. The function $f_1(p)$ and its roots shall be described in more detail in our upcoming journal paper on Maxwell Strata on SH(2). Similarly, for $\lambda \in C_2$ first conjugate time is bounded as $t_1^{C_2} = kt_1^{C_1}$. Thus globally the first conjugate time is greater or equal to the first Maxwell time. We conjecture that the cut time is equal to the first Maxwell time. This conjecture will be studied in a forthcoming work.

References

1. Enrico Le Donne. Lecture notes on sub-Riemannian geometry. Preprint, 2010.
2. N. Ja. Vilenkin. Special Functions and Theory of Group Representations (Translations of Mathematical Monographs). American Mathematical Society, revised edition, 1968.
3. Yuri L. Sachkov. Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane, ESAIM: COCV, Volume 16, Issue 04, October 2010, pp 1018–1039
4. Yuri L. Sachkov. Conjugate points in the Euler elastic problem. Journal of Dynamical and Control Systems, 14:409–439, July 2008.
5. A. A. Ardentov and Yu. L. Sachkov. Conjugate points in nilpotent sub-Riemannian problem on the Engel group, Journal of Mathematical Sciences December 2013, Volume 195, Issue 3, pp 369–390