Flows on homogeneous spaces and Diophantine approximation

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Introduction

There are several main directions in the theory of Diophantine approximations. Here are examples of some problems and open questions.

Problem 1 Given m vectors $\mathbf{y}_1,\ldots,\mathbf{y}_m \in \mathbb{R}^n$ and the non-increasing function $\psi : \mathbb{R} \to \mathbb{R}$, how many solutions $(\mathbf{p}, q)$ exist for inequality :

$$max_{1\leq i\leq m}\|q\mathbf{y}_i - \mathbf{p}\| < \psi(q)$$

where $q \in \mathbb{Z} \setminus 0, \mathbf{p} \in \mathbb{Z}^n, \mathbf{v}\| = max_{j=1,\ldots,n}|v_j|$?

Problem 2 Given non-increasing function $\psi : \mathbb{R} \to \mathbb{R}$,what is the measure of such sets $y_1,\ldots,y_m \in \mathbb{R}^n$(in the sense of the Lebesgue measure on $\mathbb{R}^{mn}$) that the inequality :

$$max_{1\leq i\leq m}\|q\mathbf{y}_i - \mathbf{p}\| < \psi(q)$$

has infinitely many solutions $(q, \mathbf{p}) \in \mathbb{Z}^{n+1}$?

Problem 3 Given non-increasing function $\psi : \mathbb{R} \to \mathbb{R}$, open and connected subset U of $\mathbb{R}^d$ and $f_1,\ldots,f_n \in C^n(U)$,$d < n$,what is the measure of such points $\mathbf{y} \in M = \{(f_1(x),\ldots,f_n(x))|x \in U\}$ (in the sense of the Lebesgue measure on U) that the inequality :

$$\|q\mathbf{y} - \mathbf{p}\| < \psi(q)$$

has infinitely many solutions $(q, \mathbf{p}) \in \mathbb{Z}^{n+1}$?

After the work [D1],[KM1],[KM2] it became clear that these issues are closely related to the behavior of some flows on homogeneous spaces.

Preliminaries

Definition 1Fix $n \in \mathbb{N}$ and consider $\Omega \stackrel{def}{=} \{$ the set of unimodular lattices in $\mathbb{R}^n\} = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ - is the space of lattices.

In the future, we will need the ability to determine whether a certain trajectory in the lattice space is bounded or not. We will understand boundedness as belonging to some compact set.

Theorem(Mahler's compactness theorem) Let $\mathfrak{F}$ be some subset of Ω. $\mathfrak{F}$ is relatively compact if and only if there is a number $\rho > 0$ such that for every lattice $\mathfrak{l} \in \mathfrak{F}$ $\quad inf_{v\in\mathfrak{l}}\|v\| \geq \rho$

Definition 2Let $\mathbf{y} \in \mathbb{R}^n$,$\mathbf{y}$ is Very Well Approximable(VWA) if for some $\epsilon > 0$ there are infinitely many $q \in \mathbb{Z}, \mathbf{p} \in \mathbb{Z}^n$ such that :

$$\|q\mathbf{y} - \mathbf{p}\|^n < \frac{1}{|q|^{1+\epsilon}}$$

Definition 3Let $\mathbf{y} \in \mathbb{R}^n$,$\mathbf{y}$ is Very Well Multiplicatively Approximable(VWMA) if for some $\epsilon > 0$ there are infinitely many $q \in \mathbb{Z}, \mathbf{p} \in \mathbb{Z}^n$ such that :

$$\prod_{i=1}^{n} |qy_i - p_i| < \frac{1}{|q|^{1+\epsilon}}$$

It is not difficult to see that VWMA-numbers are also VWA-numbers.

Definition 4 Let U be open and connected subset of $\mathbb{R}^d$,$f_1,...,f_n \in C^n(U)$,manifold $M = \{(f_1(x),\ldots,f_n(x))|x \in U\}$ is called extremal if almost all points M relative to the Lebesgue measure on U are not VWA

Main Results

Definition 4 Let $\mathbf{y} \in \mathbb{R}^n$,then $L_{\mathbf{y}} = \begin{pmatrix} 1 & \mathbf{y}^T \\ 0 & Id_n \end{pmatrix}$,where Id_n is identity $n \times n$ matrix

Definition 5 $\quad g_{\mathbf{t}} = diag(e^{t_0}, e^{-t_1},\ldots,e^{-t_n}), \mathbf{t} = (t_0,t_1,\ldots,t_n), \sum_{i=1}^n t_i = t_0$ is geodesic flow

Theorem(Dani's correspondence[D1]) If $\mathbf{y} \in \mathbb{R}^n$ is $VWMA$, then $g_{\mathbf{t}}(L_{\mathbf{y}})$ is unbounded(in sence of Mahler's compactness theorem)

Theorem(Khinchin-Groshev theorem [KM1]) Let $\psi : \mathbb{R} \to \mathbb{R}$be a non-increasing continuous function.If there are infinitely many solutions $(\mathbf{q}, p) \in \mathbb{Z}^{n+1}$ to the inequality

$$\|(\mathbf{q}, \mathbf{y}) - p\| < \psi(\|q\|^n)$$

for almost all((resp. almost no) $textbfy$ then the integral $\int_1^\infty \psi(x)dx$ diverges (resp.converges)

Theorem([KM2]) Let $f1,\ldots,f_n$ be analytic in$U$,where U is an open subset of $\mathbb{R}^d$,which together with 1 are linearly independent over $\mathbb{R}$.Then the manifold $M = \{(f_1(x),\ldots,f_n)|xU\}$ is strongly extremal

References

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KM1 D. Kleinbock and G. A. Margulis, Logarithm laws for flows on homogeneous spaces, Inv. Math. 138 (1999), 451–494.

KM2 D. Kleinbock and G. A. Margulis Flows on homogeneous spaces and Diophantine approximation on manifolds,Ann. Math. 148 (1998), 339–360.